# June 14 Math 1190 sec. 51 Summer 2017

## Section 2.4: Differentiating a Product or Quotient; Higher Order Derivatives

First derivative: $\dfrac{dy}{dx} = y' = f'(x)$

Second derivative: $\dfrac{d}{dx}\dfrac{dy}{dx} = \dfrac{d^2y}{dx^2} = y'' = f''(x)$

Third derivative: $\dfrac{d}{dx}\dfrac{d^2y}{dx^2} = \dfrac{d^3y}{dx^3} = y''' = f'''(x)$

Fourth derivative: $\dfrac{d}{dx}\dfrac{d^3y}{dx^3} = \dfrac{d^4y}{dx^4} = y^{(4)} = f^{(4)}(x)$

$n^{th}$ derivative: $\dfrac{d}{dx}\dfrac{d^{n-1}y}{dx^{n-1}} = \dfrac{d^ny}{dx^n} = y^{(n)} = f^{(n)}(x)$

# Remarks on Notation

- $\frac{d}{dx}$ can *operate* on a function to produce a new function; e.g.

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

- It's too hard to read multiple primes (say beyond 3). Parentheses **must** be used to distinguish powers from derivatives.

$$y^5 \text{ is the fifth power of } y;$$

$$y^{(5)} \text{ is the fifth derivative of } y$$

# Example

Compute the first, second, and third derivatives of $f(x) = 3x^4 + 2x^2$.

$$f'(x) = 3(4x^3) + 2(2x)$$

$$= 12x^3 + 4x$$

$$f''(x) = 12(3x^2) + 4(1)$$

$$= 36x^2 + 4$$

$$f'''(x) = 36(2x) + 0 = 72x$$

# Example

prod rule : $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)\,g'(x)$

Evaluate $F''(x)$ and $F''(2)$ where $F(x) = x^3 e^x$.

$$F'(x) = \left(\frac{d}{dx}x^3\right)e^x + x^3\left(\frac{d}{dx}e^x\right)$$

$$= 3x^2 e^x + x^3 e^x$$

$$F''(x) = \left(\frac{d}{dx}3x^2\right)e^x + 3x^2\left(\frac{d}{dx}e^x\right) + \left(\frac{d}{dx}x^3\right)e^x + x^3\left(\frac{d}{dx}e^x\right)$$

$$= 6x e^x + 3x^2 e^x + 3x^2 e^x + x^3 e^x$$

$$F''(x) = 6xe^x + 6x^2 e^x + x^3 e^x, \quad F''(2) = 6(2)e^2 + 6(2)^2 e^2 + 2^3 e^2$$

$$= 44e^2$$

# Question

Let $a$, $b$, and $c$ be nonzero constants. If $y = ax^2 + bx + c$, then $\frac{d^3y}{dx^3}$ is

(a) 0

(b) $2a + b + c$

(c) $2a$

(d) cannot be determined without knowing the values of $a$, $b$, and $c$.

$y' = 2ax + b$

$y'' = 2a$

$y''' = 0$

# Question

**True or False:** The fourth derivative of a function $y = f(x)$ is denoted by

$$\frac{dy^4}{dx^4}.$$

Should be $\frac{d^4y}{dx^4}$

# Rectilinear Motion

If the position $s$ of a particle in motion (relative to an origin) is a differentiable function $s = f(t)$ of time $t$, then the derivatives are physical quantities.

**Velocity:** is the rate of change of position with respect to time $v = f'(t)$.

**Acceleration:** is the rate of change of velocity with respect to time $a = \frac{dv}{dt} = f''(t)$.

# Galileo's Law

Galileo's law states that in a vacuum (i.e. in the absence of fluid drag), the position of any object falling near the Earth's surface, subject only to gravity, is proportional to the square of the time elapsed. Mathematically, position $s$ satisfies

$$s = -ct^2.$$

Show that this statement is equivalent to saying that the acceleration due to gravity is constant.

Velocity $v = \frac{ds}{dt} = -2ct$

acceleration $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -2c$ a constant

# Question

A particle moves along the $x$-axis so that its position relative to the origin satisfies $s = t^3 - 4t^2 + 5t$. Determine the acceleration of the particle at time $t = 1$.

(a) $a(1) = 0$

(b) $a(1) = -2$

(c) $a(1) = 6t - 8$

(d) $a(1) = 3t^2 - 8t + 5$

# Section 2.5: The Derivative of the Trigonometric Functions

We wish to arrive at derivative rules for each of the six trigonometric functions.

Recall the limits from before

is a this limit

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \quad\text{and}\quad \lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0$$

The equation $\frac{\sin\theta}{\theta}=1$ is <u>never</u> true.

$$\frac{d}{dx}\sin(x) = \cos(x) \quad \text{and} \quad \frac{d}{dx}\cos(x) = -\sin(x)$$

We'll prove the first (the second is left as an exercise).

Recall $\sin(A+B) = \sin A \cos B + \sin B \cos A$

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) - \sin(x) + \sin(h)\cos(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x)\left(\cos(h) - 1\right) + \cos(x)\sin(h)}{h}$$

$$= \lim_{h \to 0} \left( \frac{\sin(x)\left(\cos(h) - 1\right)}{h} + \frac{\cos(x)\sin(h)}{h} \right)$$

$$= \lim_{h \to 0} \left( \sin(x)\left( \frac{\cos(h) - 1}{h} \right) + \cos(x)\left( \frac{\sin(h)}{h} \right) \right)$$

$$= \sin(x) \cdot 0 + \cos(x) \cdot 1$$

$$= \cos(x)$$

That is, $\frac{d}{dx}\sin(x) = \cos(x)$

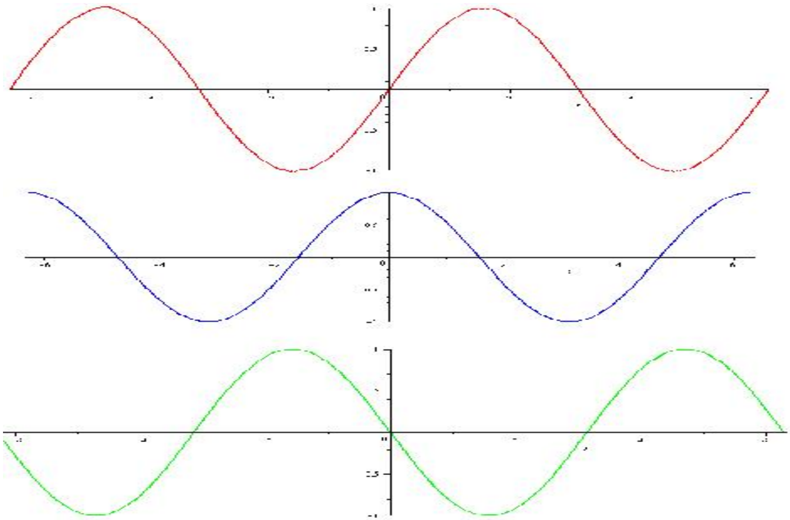

Figure: Graphs of $y = \sin x$, $y = \cos x$, $y = -\sin x$ (from top to bottom).

## Examples: Evaluate the derivative.

(a) $\dfrac{d}{dx}(\sin x + 4\cos x)$ $= \dfrac{d}{dx}\sin x + 4\dfrac{d}{dx}\cos x$

$= \cos x + 4(-\sin x)$

$= \cos x - 4\sin x$

(b) $\dfrac{d}{d\theta}\theta^4 \sin\theta$ $= \left(\dfrac{d}{d\theta}\theta^4\right)\sin\theta + \theta^4\left(\dfrac{d}{d\theta}\sin\theta\right)$

$= 4\theta^3\sin\theta + \theta^4\cos\theta$

Use the fact that $\tan x = \sin x / \cos x$ to determine the derivative rule for the tangent.

$$\frac{d}{dx}\frac{f}{g} = \frac{f'g - fg'}{(g)^2}$$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\left(\frac{d}{dx}\sin x\right)\cos x - \sin x\left(\frac{d}{dx}\cos x\right)}{(\cos x)^2}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

# Six Trig Function Derivatives

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x,$$

$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\cot x = -\csc^2 x,$$

$$\frac{d}{dx}\sec x = \sec x\tan x, \qquad \frac{d}{dx}\csc x = -\csc x\cot x$$

# Question

Use the known derivatives $\dfrac{d}{dx}\tan x = \sec^2 x$ and $\dfrac{d}{dx}\sec x = \sec x \tan x$ to evaluate the derivative of $f$ where

$$f(x) = \tan x \sec x$$

(a) $f'(x) = \sec^2 x \sec x \tan x$

(b) $f'(x) = \sec^3 x + \sec x \tan^2 x$ product rule

(c) $f'(x) = \cot x \sec x + \tan x \csc x$

(d) $f'(x) = \sec x$

# Example

Find the equation of the line tangent to the graph of $y = \csc x$ at the point $(\pi/6, 2)$.

we need $m_{tan}$. $m_{tan} = \frac{dy}{dx}$ @ $x = \frac{\pi}{6}$

$$\frac{dy}{dx} = -\csc x \cot x \quad \text{so} \quad m_{tan} = -\csc\frac{\pi}{6}\cdot\cot\frac{\pi}{6} = -2\cdot\sqrt{3}$$

$y - y_0 = m(x - x_0)$

$$y - 2 = -2\sqrt{3}\left(x - \frac{\pi}{6}\right)$$

$$y = -2\sqrt{3}\,x + \frac{2\sqrt{3}\pi}{6} + 2$$

$$\boxed{y = -2\sqrt{3}\,x + \frac{\sqrt{3}\pi}{3} + 2}$$

# Section 3.1: The Chain Rule

Suppose we wish to find the derivative of $f(x) = (x^2 + 2)^2$.

we can expand the square $f(x) = x^4 + 4x^2 + 4$

so $f'(x) = 4x^3 + 8x + 0$

Now suppose we want to differentiate $g(x) = (x^2 + 2)^{10}$. How about $F(x) = \sqrt{x^2 + 2}$?

We could expand g, but it's a lot of work, we have no way of finding $F'(x)$ with our current rules.

# Example of Compositions

Find functions $f(u)$ and $g(x)$ such that

$$F(x) = \sqrt{x^2+2} = (f \circ g)(x).$$

If $f(u) = \sqrt{u}$ and $g(x) = x^2+2$

then $F(x) = (f \circ g)(x)$

Check: $(f \circ g)(x) = f(g(x))$

$= f(x^2+2) = \sqrt{x^2+2} = F(x)$ ✓

# Example of Compositions

Find functions $f(u)$ and $g(x)$ such that

$$F(x) = \cos\left(\frac{\pi x}{2}\right) = (f \circ g)(x).$$

Take $f(u) = \cos u$ and $g(x) = \frac{\pi}{2} x$

Check: $(f \circ g)(x) = f(g(x)) = f\left(\frac{\pi}{2} x\right)$

$= \cos\left(\frac{\pi}{2} x\right) = F(x)$ ✓

# Theorem: Chain Rule

Suppose $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$. Then the composite function

$$F = f \circ g$$

is differentiable at $x$ and

$$\frac{d}{dx}F(x) = \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

In Liebniz notation: if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

# Example

Determine any inside and outside functions and find the derivative.

(a) $F(x) = \sin^2 x$ $= (\sin x)^2$

outside $f(u) = u^2$ and inside $u = g(x) = \sin x$

$f'(u) = 2u$  $g'(x) = \cos x$

$F'(x) = f'(g(x))\,g'(x)$

so

$F'(x) = 2\sin x \cdot \cos x$

that's $2u$ where $u = \sin x$

(b) $F(x) = e^{x^4-5x^2+1} = \exp(x^4-5x^2+1)$

Here, $f(u) = e^u$ and $u = g(x) = x^4-5x^2+1$

$f'(u) = e^u$ $g'(x) = 4x^3-10x$

$F'(x) = f'(g(x))\,g'(x)$

$= e^{x^4-5x^2+1}\cdot(4x^3-10x)$

$= (4x^3-10x)\,e^{x^4-5x^2+1}$

# Question

Find $G'(\theta)$ where $\quad G(\theta) = \cos\left(\frac{\pi\theta}{2} - \frac{\pi}{4}\right)$

Use inside $\quad u = \frac{\pi\theta}{2} - \frac{\pi}{4}, \quad$ and outside $\quad G(u) = \cos u$

$u' = \frac{\pi}{2} \qquad G'(u) = -\sin u$

(a) $\quad G'(\theta) = -\frac{\pi}{2}\sin\left(\frac{\pi\theta}{2} - \frac{\pi}{4}\right)$

(b) $\quad G'(\theta) = -\sin\left(\frac{\pi\theta}{2}\right) - \sin\left(\frac{\pi}{4}\right)$

$u = \frac{\pi}{2}\theta - \frac{\pi}{4}$

(c) $\quad G'(\theta) = -\frac{\pi\theta}{2}\sin\left(\frac{\pi\theta}{2} - \frac{\pi}{4}\right)$

$u' = \frac{\pi}{2} \cdot 1 - 0$

(d) $\quad G'(\theta) = -\frac{\pi}{2}\sin\left(\frac{\pi\theta}{2}\right)$

# The power rule with the chain rule

If $u = g(x)$ is a differentiable function and $n$ is any integer, then

$$\frac{d}{dx}u^n = \underbrace{nu^{n-1}}_{f'(g(x))} \cdot \underbrace{\frac{du}{dx}}_{g'(x)}.$$

where $f(u) = u^n$

Evaluate: $\frac{d}{dx}e^{7x} = \frac{d}{dx}(e^x)^7$

here $f(u) = u^7$ so $f'(u) = 7u^6$

$g(x) = e^x$ so $g'(x) = e^x$

So $\frac{d}{dx}e^{7x} = \frac{d}{dx}(e^x)^7 =$

$$= 7(e^x)^6 \cdot e^x$$

$$= 7e^{6x} \cdot e^x$$

$$= 7e^{6x+x} = 7e^{7x}$$

Note If $f(u) = e^u$ and $g(x) = 7x$

$f'(u) = e^u$ and $g'(x) = 7$

so $\frac{d}{dx} e^{7x} = e^{7x} \cdot 7 = 7e^{7x}$

# Question

Use the power rule with the chain rule to find the derivative of $f(x) = \cos^2 x$.

$\frac{d}{dx} u^2 = 2u \frac{du}{dx}$

(a) $f'(x) = -\sin^2 x$

(b) $f'(x) = 2\cos x$

(c) $f'(x) = -2\cos x \sin x$

(d) $f'(x) = -\sin^2 x \cos x$

# Questions

Consider the composition $f(x) = e^{\sin x}$.

Which pair could be the inside and outside functions in this composition?

(a) inside $e^x$ and outside $\sin x$ $\rightarrow$ $\sin(e^x)$

(b) inside $\ln x$ and outside $\sin x$

(c) inside $\sin x$ and outside $e^x$ $\rightarrow$ $e^{\sin x}$

(d) inside $\sin x$ and outside $\ln x$

# Questions

Use the chain rule to find the derivative $\frac{d}{dx} e^{\sin x}$.

(a) $\sin x \; e^{\sin x - 1}$

(b) $\cos x \; e^{\sin x - 1}$

(c) $\cos x \; e^{\sin x}$

(d) $e^{\sin x}$

# Questions

If $f(x) = e^{\sin x}$, the value of $f(0)$ is

(a) $f(0) = 0$

$\sin(0) = 0$ and $e^0 = 1$

(b) $f(0) = 1$

(c) $f(0) = e$

(d) $f(0)$ can't be determined without more information.

# Questions

Find the equation of the line tangent to the graph of $f(x) = e^{\sin x}$ at the point $(0, f(0))$.

$f'(0) = \cos(0)\, e^{\sin(0)} = 1 \cdot 1 = 1$

and $(0, f(0)) = (0, 1)$.

(a) $y = x + 1$

(b) $y = 1$

(c) $y = x - 1$

(d) $y = ex + 1$

# Multiple Compositions

The chain rule can be iterated to account for multiple compositions. For example, suppose $f$, $g$, $h$ are appropriately differentiable, then

$$\frac{d}{dx}(f \circ g \circ h)(x) = \frac{d}{dx} f(g(h(x))) = f'(g(h(x))) g'(h(x))\, h'(x)$$

Note that the outermost function is $f$, and its inner function is a composition $g(h(x))$.

So the derivative of the outer function evaluated at the inner is $f'(g(h(x))$ which is multiplied by the derivative of the inner function—**itself based on the chain rule**—$g'(h(x))h'(x)$.

# Example

Evaluate the derivative $\dfrac{d}{dt}\tan^2\left(\dfrac{1}{3}t^3\right) = \dfrac{d}{dt}\left(\tan\left(\frac{1}{3}t^3\right)\right)^2$

Here $f(u) = u^2$ $\quad u = \tan\left(\frac{1}{3}t^3\right) = \tan(v)$ where $v = \frac{1}{3}t^3$

$f'(u) = 2u$ $\quad \dfrac{d}{dv}\tan(v) = \sec^2(v)$ $\quad$ and $\dfrac{dv}{dt} = \frac{1}{3}(3t^2) = t^2$

so

$$\frac{d}{dt}\tan^2\left(\tfrac{1}{3}t^3\right) = 2\tan\left(\tfrac{1}{3}t^3\right)\cdot\sec^2\left(\tfrac{1}{3}t^3\right)\cdot t^2$$

$$= 2t^2\tan\left(\tfrac{1}{3}t^3\right)\sec^2\left(\tfrac{1}{3}t^3\right)$$

# Exponential of Base $a$

Let $a > 0$ with $a \neq 1$. By properties of logs and exponentials

$$a^x = e^{(\ln a)x}.$$

$(\ln a)x$ is $\ln a$ times $x$

Recall $x \ln a = \ln a^x$

so $e^{x \ln a} = e^{\ln a^x} = a^x$

$\frac{d}{dx} x \ln a = \ln a$

$\frac{d}{dx} e^{(\ln a)x} = e^{(\ln a)x} \cdot \ln a$

$= a^x \cdot \ln a$

**Theorem: (Derivative of $y = a^x$)** Let $a > 0$ and $a \neq 1$. Then

$$\frac{d}{dx} a^x = a^x \ln a$$

# Example

Evaluate

(a) $\dfrac{d}{dx}4^x$ $= 4^x \ln 4$

$f(u) = 2^u$

$f'(u) = 2^u \ln 2$

(b) $\dfrac{d}{dx}2^{\cos x}$ $= 2^{\cos x} \ln 2 \cdot (-\sin x)$

$u = \cos x$

$= -\ln 2 \sin x \; 2^{\cos x}$

$u' = -\sin x$

# Section 3.2: Implicit Differentiation; Derivatives of the Inverse Trigonometric Functions

The chain rule states that for a differentiable composition $f(g(x))$

$$\frac{d}{dx}\, f(g(x)) = f'(g(x))g'(x)$$

For $y = f(u)$ and $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

# Example

Assume $f$ is a differentiable function of $x$. Find an expression for the derivative:

Outside $u^2$ $\frac{d}{du}u^2 = 2u$

Inside $f(x)$ $\frac{d}{dx}f(x) = f'(x)$

$$\frac{d}{dx}\left(f(x)\right)^2 = 2f(x) \cdot f'(x)$$

Outside $\tan(u)$ $\frac{d}{du}\tan u = \sec^2 u$

Inside $f(x)$ $\frac{d}{dx}f(x) = f'(x)$

$$\frac{d}{dx}\tan\left(f(x)\right) = \sec^2\left(f(x)\right) \cdot f'(x)$$

# Example

Suppose we know that $y = f(x)$ for some differentiable function (but we don't know exactly what $f$ is). Find an expression for the derivative.

$$\frac{d}{dx}\sqrt{y} = \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx}$$

outside $\sqrt{y}$, $\frac{d}{dy}\sqrt{y} = \frac{1}{2\sqrt{y}}$

inside $y$, $\frac{d}{dx}y = \frac{dy}{dx}$

$$\frac{d}{dx}x^2y^2 = \left(\frac{d}{dx}x^2\right)y^2 + x^2\left(\frac{d}{dx}y^2\right)$$

$$= 2xy^2 + x^2\left(2y \cdot \frac{dy}{dx}\right) = 2xy^2 + 2x^2y\frac{dy}{dx}$$

# Question

Assuming that $y$ is some differentiable function of $x$ with derivative $\frac{dy}{dx}$, the derivative of $y^3$ is

(a) $\dfrac{d}{dx}y^3 = 3y^2$

(b) $\dfrac{d}{dx}y^3 = 3y^2\,\frac{dy}{dx}$

(c) $\dfrac{d}{dx}y^3 = 3\left(\frac{dy}{dx}\right)^2$

# Example

Consider the simple example $y = x^2$. Compute $\frac{d}{dx}y^3$.

If $y = x^2$, then $y^3 = (x^2)^3 = x^6$

so $\frac{d}{dx} y^3 = \frac{d}{dx} x^6 = 6x^5$

# Example

Consider the simple example $y = x^2$ so that $\frac{dy}{dx} = 2x$. Compute each of

(a) $3y^2$ $= 3(x^2)^2 = 3x^4$

(b) $3y^2\dfrac{dy}{dx}$ $= 3(x^2)^2 \cdot (2x) = 3x^4(2x) = 6x^5$

(c) $3\left(\dfrac{dy}{dx}\right)^2$ $= 3(2x)^2 = 3(4x^2) = 12x^2$

# Implicitly defined functions

A relation—an equation involving two variables $x$ and $y$—such as

$$x^2 + y^2 = 16 \quad \text{or} \quad (x^2 + y^2)^3 = x^2$$

**implies** that $y$ is defined to be one or more functions of $x$.

for example $x^2 + y^2 = 16 \Rightarrow y^2 = 16 - x^2$

$\Rightarrow y = \sqrt{16 - x^2}$ or $y = -\sqrt{16 - x^2}$

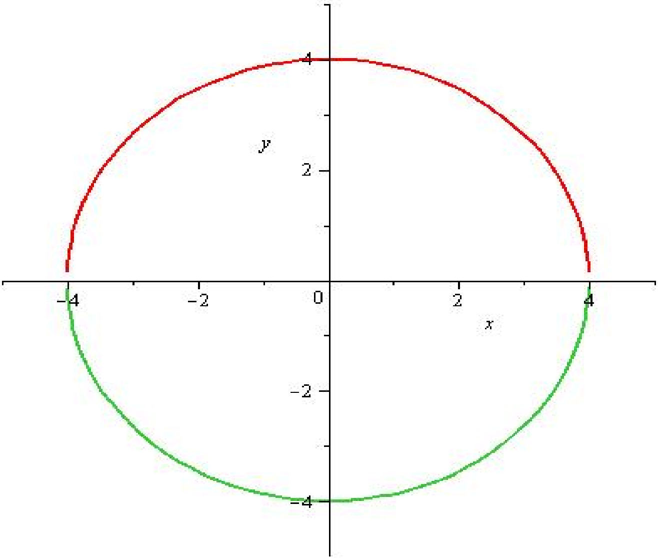


Figure: $x^2 + y^2 = 16$

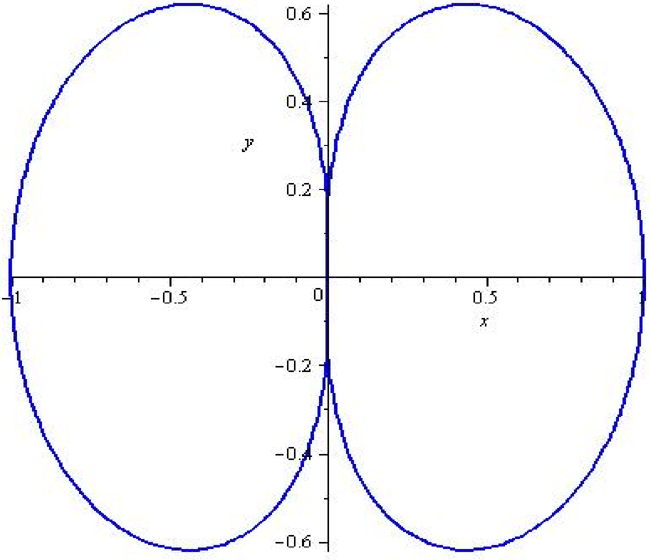

Figure: $(x^2 + y^2)^3 = x^2$

# Explicit -vs- Implicit

A function is defined **explicitly** when given in the form

$$y = f(x).$$

e.g. $y = \tan x$ or $y = e^{\sin x}$

A function is defined *implicitly* when it is given as a relation

$$F(x, y) = C,$$

for constant $C$.

e.g. $(x^2+y^2)^3 - x^2 = 0$ , or $y \ln y = x e^y + \cos x$

# Implicit Differentiation

Since $x^2 + y^2 = 16$ *implies* that $y$ is a function of $x$, we can consider it's derivative.

$$\text{Find } \frac{dy}{dx} \text{ given } \quad x^2 + y^2 = 16.$$

Take $\frac{d}{dx}$ of both sides of the relation.

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(16) \Rightarrow \frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 0$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0$$ Isolate $\frac{dy}{dx}$

$$2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} \Rightarrow \boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

Show that the same result is obtained knowing

$$y = \sqrt{16 - x^2} \quad \text{or} \quad y = -\sqrt{16 - x^2}.$$

I'll do the first and leave the second as an exercise.

outside $\sqrt{u}$ and inside $u = 16 - x^2$

$$\frac{d}{du}\sqrt{u} = \frac{1}{2\sqrt{u}} \quad \text{and} \quad \frac{d}{dx}(16 - x^2) = -2x$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{16-x^2}} \cdot (-2x) = \frac{-\cancel{2}x}{\cancel{2}\sqrt{16-x^2}}$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{16-x^2}}$$

But $\sqrt{16-x^2} = y$ so

$$\frac{dy}{dx} = \frac{-x}{y}.$$