June 18 Math 2254 sec 001 Summer 2015

Section 7.3 Trigonometric Substitution

Compare the two integrals:

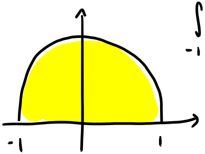
$$\int x\sqrt{1-x^2}\,dx, \quad \text{and} \quad \int \sqrt{1-x^2}\,dx.$$

$y = \sqrt{1-x^2} \Rightarrow y^2 = 1-x^2$

$\Rightarrow x^2+y^2 = 1$

Using some simple geometry

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi}{2}.$$



$\int_{-1}^{1}\sqrt{1-x^2}\,dx$ = Area

radius $r=1$

Area $= \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$

It's clear from geometry, but **NOT AT ALL** from any algebra where a factor of π comes from!

A different sort of substitution

$$\int \sqrt{1-x^2}\,dx$$

Consider the new variable θ assuming $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ defined by

$$x = \sin\theta, \quad \text{so that} \quad dx = \cos\theta\, d\theta$$

$$1-x^2 = 1-(\sin\theta)^2 = 1-\sin^2\theta = \cos^2\theta$$

$$\sqrt{1-x^2} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta \quad \text{for } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

(quads I or IV)

Replacing all terms in the integral

$$\int \sqrt{1-x^2}\,dx = \int \cos\theta \cdot \cos\theta \, d\theta$$

$$= \int \cos^2\theta \, d\theta$$

$x = \sin\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$= \int \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

so $\theta = \sin^{-1} x$

$$= \int \frac{1}{2}\, d\theta + \int \frac{1}{2}\cos 2\theta \, d\theta$$

also

$$= \frac{1}{2}\theta + \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta + C$$

$\sin 2\theta = 2\sin\theta\cos\theta$

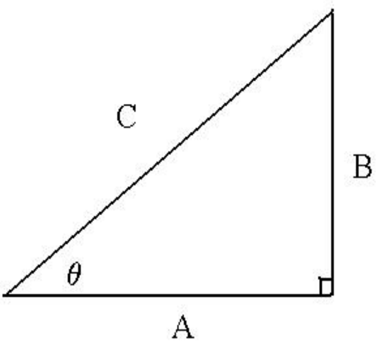
$$= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$

$$= \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2} + C$$

Trigonometric Substitution

May be useful when we see terms in an integral that look like (a is a constant)

$$\sqrt{a^2 - x^2}, \quad \sqrt{x^2 - a^2}, \quad \text{or} \quad \sqrt{a^2 + x^2}.$$



$C^2 = A^2 + B^2$

$C = \sqrt{A^2 + B^2}$

hypotenuse root of sum of squares

$A^2 = C^2 - B^2$

$A = \sqrt{C^2 - B^2}$

$B^2 = C^2 - A^2$

$B = \sqrt{C^2 - A^2}$

leg root of Difference w/ hypotenuse first

Figure: Trig Substitution Motivated by the Pythagorean Theorem

Substitution for the form $\sqrt{a^2 - x^2}$

$$x = a\sin\theta$$

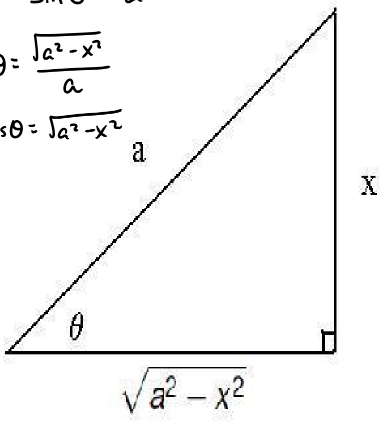
$$dx = a\cos\theta\, d\theta$$

$$\sqrt{a^2 - x^2} = a\cos\theta$$

$\sin\theta = \frac{x}{a} \Rightarrow x = a\sin\theta$

$\cos\theta = \frac{\sqrt{a^2 - x^2}}{a}$

$a\cos\theta = \sqrt{a^2 - x^2}$



Substitution for the form $\sqrt{x^2 - a^2}$

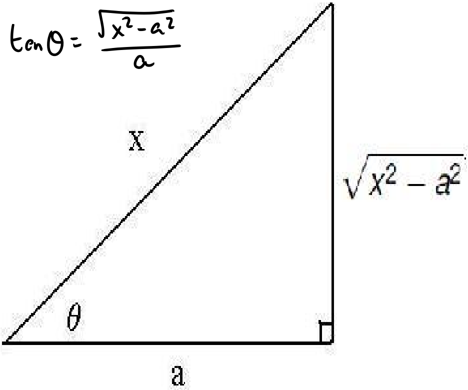
$$\sec\theta = \frac{x}{a} \Rightarrow x = a\sec\theta$$

$$\tan\theta = \frac{\sqrt{x^2 - a^2}}{a}$$

$$x = a\sec\theta$$

$$dx = a\sec\theta\tan\theta\, d\theta$$

$$\sqrt{x^2 - a^2} = a\tan\theta$$



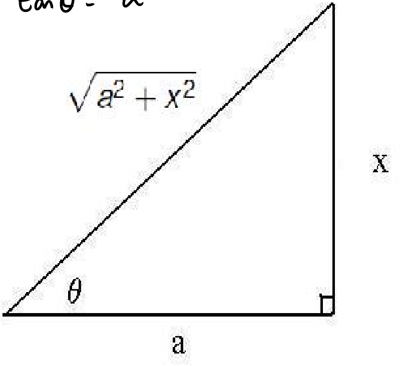
Substitution for the form $\sqrt{a^2+x^2}$

$\tan\theta = \dfrac{x}{a} \Rightarrow x = a\tan\theta$

$$x = a\tan\theta$$

$$dx = a\sec^2\theta\,d\theta$$

$$\sqrt{x^2+a^2} = a\sec\theta$$



$\sec\theta = \dfrac{\sqrt{a^2+x^2}}{a}$

We'll assume that θ is in an appropriate interval–e.g. $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ for the substitution $x = a\tan\theta$

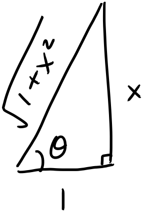
Evaluate The Integral

$\sqrt{a^2+x^2}$ where $a = 1$

(a) $$\int \frac{dx}{1+x^2} = \int \frac{dx}{\left(\sqrt{1+x^2}\right)^2}$$

$$= \int \frac{\sec^2\theta \, d\theta}{(\sec\theta)^2}$$

$$= \int \frac{\sec^2\theta \, d\theta}{\sec^2\theta}$$



$$\frac{x}{1} = \tan\theta$$

$$x = \tan\theta$$

$$dx = \sec^2\theta \, d\theta$$

$$\frac{\sqrt{1+x^2}}{1} = \sec\theta$$

$$= \int d\theta$$

$x = \tan\theta \Rightarrow$

$\theta = \tan^{-1} x$

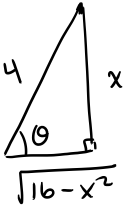
$$= \theta + C$$

$$= \tan^{-1} x + C$$

Evaluate The Integral

$\sqrt{a^2 - x^2}$ where $a = 4$

(b) $\displaystyle\int \frac{\sqrt{16 - x^2}}{x^2}\, dx$



$$\int \frac{\sqrt{16-x^2}}{x^2}\, dx = \int \frac{4\cos\theta}{(4\sin\theta)^2}\, 4\cos\theta\, d\theta$$

$$\frac{x}{4} = \sin\theta$$

$$x = 4\sin\theta$$

$$dx = 4\cos\theta\, d\theta$$

$$= \int \frac{16\cos^2\theta}{16\sin^2\theta}\, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$\frac{\sqrt{16-x^2}}{4} = \cos\theta \Rightarrow \sqrt{16-x^2} = 4\cos\theta$$

$$= \int \cot^2\theta \, d\theta$$

$$= \int (\csc^2\theta - 1) \, d\theta$$

$$= -\cot\theta - \theta + C$$

$$= -\frac{\sqrt{16-x^2}}{x} - \sin^{-1}\left(\frac{x}{4}\right) + C$$

$\cot^2\theta + 1 = \csc^2\theta$

$\sin\theta = \frac{x}{4} \Rightarrow$

$\theta = \sin^{-1}\left(\frac{x}{4}\right)$

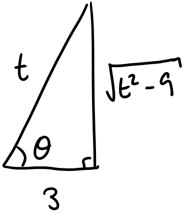
From the triangle

$\cot\theta = \frac{\sqrt{16-x^2}}{x}$

Evaluate The Integral

$\sqrt{t^2 - a^2}$ where $a = 3$

(c) $\displaystyle\int \frac{dt}{t^2\sqrt{t^2-9}}$



$\displaystyle\int \frac{dt}{t^2\sqrt{t^2-9}}$

$\displaystyle = \int \frac{3\sec\theta\tan\theta\, d\theta}{(3\sec\theta)^2\, 3\tan\theta}$

$\frac{t}{3} = \sec\theta \Rightarrow t = 3\sec\theta$

$dt = 3\sec\theta\tan\theta\, d\theta$

$\frac{\sqrt{t^2-9}}{3} = \tan\theta \Rightarrow \sqrt{t^2-9} = 3\tan\theta$

$$= \int \frac{\cancel{3}\cancel{\sec\theta}\,\cancel{\tan}\theta\, d\theta}{9\sec^{\cancel{2}}\theta \cdot \cancel{3}\,\cancel{\tan}\theta}$$

$$= \int \frac{d\theta}{9\sec\theta} = \frac{1}{9}\int \frac{d\theta}{\sec\theta}$$

$$= \frac{1}{9}\int \cos\theta\, d\theta$$

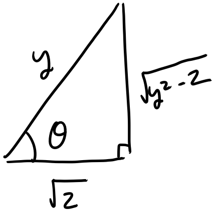
$$= \frac{1}{9}\sin\theta + C$$

$$= \frac{1}{9}\,\frac{\sqrt{t^2-9}}{t} + C$$

From the triangle $\sin\theta = \frac{opp}{hyp}$

Evaluate The Integral (other half integer roots)

(d) $\displaystyle\int \frac{dy}{(y^2-2)^{3/2}} = \int \frac{dy}{\left(\sqrt{y^2-2}\right)^3}$



$\dfrac{y}{\sqrt{2}} = \sec\theta$

$y = \sqrt{2}\sec\theta$

$dy = \sqrt{2}\sec\theta\tan\theta\, d\theta$

$\dfrac{\sqrt{y^2-2}}{\sqrt{2}} = \tan\theta \Rightarrow \sqrt{y^2-2} = \sqrt{2}\tan\theta$

$\displaystyle = \int \frac{\sqrt{2}\sec\theta\tan\theta\, d\theta}{\left(\sqrt{2}\tan\theta\right)^3}$

$\displaystyle = \int \frac{\sqrt{2}\sec\theta\tan\theta\, d\theta}{2\sqrt{2}\tan^3\theta}$

$$= \int \frac{\sec\theta}{2\tan^2\theta}\, d\theta$$

$$= \frac{1}{2} \int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\, d\theta = \frac{1}{2} \int \frac{\cos^2\theta}{\cos\theta\,\sin^2\theta}\, d\theta$$

$$= \frac{1}{2} \int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

Let $u = \sin\theta$ $\quad du = \cos\theta\, d\theta$

$$= \frac{1}{2} \int \frac{du}{u^2} = \frac{1}{2} \int u^{-2}\, du$$

$$= \frac{1}{2}\left(\frac{u^{-1}}{-1}\right) + C$$

$$= \frac{-1}{2u} + C$$

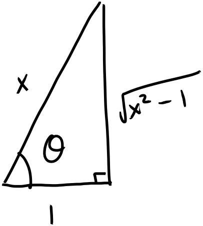
$$= \frac{-1}{2\sin\theta} + C$$

$$\int \frac{dy}{(y^2-2)^{3/2}} = \frac{-1}{2\left(\frac{\sqrt{y^2-2}}{y}\right)} + C$$

$$= \frac{-y}{2\sqrt{y^2-2}} + C$$

Evaluate The Integral

(e) $\int \frac{x^2}{\sqrt{x^2-1}}\,dx$



$= \int \frac{(\sec\theta)^2 \sec\theta \tan\theta \, d\theta}{\tan\theta}$

$\sec\theta = \frac{x}{1} \Rightarrow x = \sec\theta$

$dx = \sec\theta \tan\theta \, d\theta$

$= \int \sec^3\theta \, d\theta$

$\tan\theta = \frac{\sqrt{x^2-1}}{1}$

$\sqrt{x^2-1} = \tan\theta$

*

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\ln\left|x + \sqrt{x^2-1}\right| + C$$

$*$ $\int \sec^3\theta \, d\theta = \int \sec\theta \sec^2\theta \, d\theta$

$u = \sec\theta \quad du = \sec\theta\tan\theta \, d\theta$

Int by parts

$v = \tan\theta \quad dv = \sec^2\theta \, d\theta$

$$= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta \, d\theta$$

$$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1) \, d\theta$$

add $\int \sec^3\theta \, d\theta$

$$= \sec\theta\tan\theta - \int \sec^3\theta \, d\theta + \int \sec\theta \, d\theta$$

$$2\int \sec^3\theta \, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + K$$

$$\int \sec^3\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$