June 19 Math 1190 sec. 51 Summer 2017

Section 3.2: Implicit Differentiation; Derivatives of the Inverse Trigonometric Functions

We recall the chain rule for a differentiable composition $f(g(x))$

$$\frac{d}{dx}\, f(g(x)) = f'(g(x))g'(x)$$

For $y = f(u)$ and $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

Explicit -vs- Implicit

We also defined **implicitly defined functions** as functions that are implied by a relation

$$F(x, y) = C$$

for constant C.

We can contrast these with **explicitly** defined function given by a defining equation such as

$$y = f(x).$$

Implicit Differentiation

$$\text{Find } \frac{dy}{dx} \text{ given } \quad x^2 - 3xy + y^2 = y.$$

Take $\frac{d}{dx}$ of both sides

product

$$\frac{d}{dx}\left(x^2 - 3xy + y^2\right) = \frac{d}{dx} y$$

$$2x - 3\left(1 \cdot y + x \cdot \frac{dy}{dx}\right) + 2y \cdot \frac{dy}{dx} = \frac{dy}{dx}$$

Isolate $\frac{dy}{dx}$

$$2x - 3y - 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = \frac{dy}{dx}$$

Move $\frac{dy}{dx}$ terms to the left, all others to the right

$$-3x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = -2x + 3y$$

$$(-3x + 2y - 1)\frac{dy}{dx} = -2x + 3y$$

$$\boxed{\frac{dy}{dx} = \frac{-2x+3y}{-3x+2y-1}}$$

$ax = b$ to isolate x divide by a.

Finding a Derivative Using Implicit Differentiation:

- Take the derivative of both sides of an equation with respect to the independent variable.
- Use all necessary rules for differenting powers, products, quotients, trig functions, exponentials, compositions, etc.
- Remember the chain rule for each term involving the dependent variable (e.g. mult. by $\frac{dy}{dx}$ as required).
- Use necessary algebra to isolate the desired derivative $\frac{dy}{dx}$.

Example

Find $\frac{dy}{dx}$.

the derivative of Sin(x+y)

$$\sin(x+y) = 2x$$

$$\frac{d}{dx}\sin(x+y) = \frac{d}{dx}2x \Rightarrow \cos(x+y)\cdot\left(\frac{d}{dx}(x+y)\right) = 2$$

$$\cos(x+y)\cdot\left(1+\frac{dy}{dx}\right) = 2$$

$$\cos(x+y)\cdot 1 + \cos(x+y)\frac{dy}{dx} = 2$$

$$\cos(x+y)\frac{dy}{dx} = 2-\cos(x+y)$$

$$\boxed{\frac{dy}{dx} = \frac{2-\cos(x+y)}{\cos(x+y)}}$$

Example

Find the equation of the line tangent to the graph of $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

We need the slope m_{tan}. $m_{tan} = \frac{dy}{dx}$ @ the point $(3,3)$.

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}\left(6xy\right)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 6\left(1\cdot y + x\cdot\frac{dy}{dx}\right)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

$$\cancel{3}\left(y^2 - 2x\right)\frac{dy}{dx} = \cancel{3}(2y - x^2)$$

$$(y^2 - 2x)\frac{dy}{dx} = 2y - x^2$$

$$\boxed{\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}}$$

at the point $(3,3)$,

$$m_{tan} = \frac{2(3) - 3^2}{3^2 - 2(3)} = \frac{6-9}{9-6} = \frac{-3}{3} = -1$$

Using point slope form

$$y - 3 = -1(x - 3)$$

$$y - 3 = -x + 3$$

$$\boxed{y = -x + 6}$$

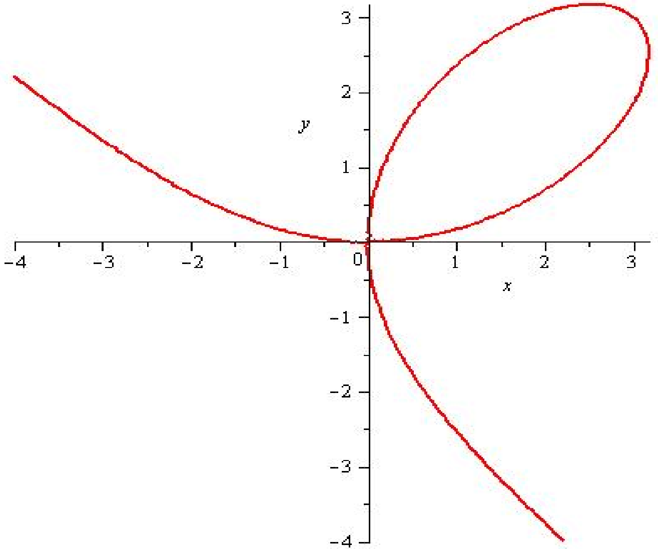


Figure: Folium of Descartes $x^3 + y^3 = 6xy$

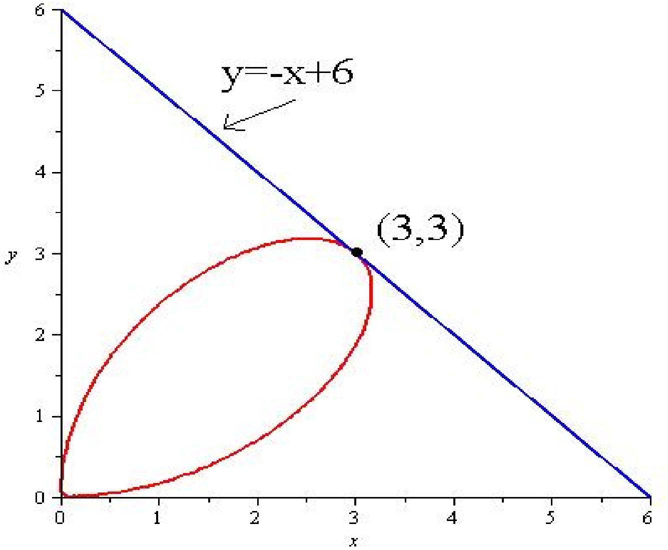


Figure: Folium of Descartes with tangent line at $(3, 3)$

The Power Rule: Rational Exponents

Let $y = x^{p/q}$ where p and q are integers. This can be written implicitly as

$$y^q = x^p.$$

Find $\frac{dy}{dx}$.

* Note: ① $\frac{x^p}{y^q} = 1$ and ② $\frac{y}{x} = \frac{x^{p/q}}{x} = x^{\frac{p}{q}-1}$

From $y^q = x^p$

$$\frac{d}{dx} y^q = \frac{d}{dx} x^p$$

$$q y^{q-1} \cdot \frac{dy}{dx} = p x^{p-1}$$

$$\frac{dy}{dx} = \frac{p\,x^{p-1}}{q\,y^{q-1}} = \frac{p}{q}\,\frac{x^p x^{-1}}{y^q y^{-1}}$$

$$\frac{dy}{dx} = \frac{p}{q}\,\frac{x^p\,y}{y^q\,x} = \frac{p}{q}\left(\frac{x^p}{y^q}\right)\left(\frac{y}{x}\right)$$

$*$ $$\frac{dy}{dx} = \frac{p}{q}(1)\,x^{\frac{p}{q}-1}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{p}{q}\,x^{\frac{p}{q}-1}$$

the regular power rule!

The Power Rule: Rational Exponents

Theorem: If r is any rational number, then when x^r is defined, the function $y = x^r$ is differentiable and

$$\frac{d}{dx}x^r = rx^{r-1}$$

for all x such that x^{r-1} is defined.

e.g. $\frac{d}{dx} x^{1/2} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

$\frac{d}{dx} x^{8/9} = \frac{8}{9} x^{8/9 - 1} = \frac{8}{9} x^{-1/9}$

Examples

Evaluate

(a) $\frac{d}{dx}\sqrt[4]{x}$ $= \frac{d}{dx} x^{1/4} = \frac{1}{4} x^{\frac{1}{4}-1} = \frac{1}{4} x^{-3/4}$

(b) $\frac{d}{dv}\csc(\sqrt{v})$ $= -\csc(\sqrt{v})\cot(\sqrt{v}) \cdot \frac{1}{2\sqrt{v}}$

$= \frac{-\csc(\sqrt{v})\cot(\sqrt{v})}{2\sqrt{v}}$

outside

$\csc(u)$ $\frac{d}{du}\csc u = -\csc u \cot u$

Inside

$u = \sqrt{v}$ $\frac{du}{dv} = \frac{1}{2\sqrt{v}}$

Question

Find $f'(x)$ where $f(x) = \sqrt[5]{x^7}$.

$f(x) = x^{7/5}$

(a) $f'(x) = \frac{7}{5}x^{2/5}$

(b) $f'(x) = \frac{5}{7}x^{-2/7}$

(c) $f'(x) = \frac{1}{5}\left(x^7\right)^{-4/5}$

(d) $f'(x) = \sqrt[5]{7x^6}$

Inverse Functions

Suppose $y = f(x)$ and $x = g(y)$ are inverse functions—i.e. $(g \circ f)(x) = g(f(x)) = x$ for all x in the domain of f.

Theorem: Let f be differentiable on an open interval containing the number x_0. If $f'(x_0) \neq 0$, then g is differentiable at $y_0 = f(x_0)$. Moreover

$$\frac{d}{dy}g(y_0) = g'(y_0) = \frac{1}{f'(x_0)}.$$

Note that this refers to a pair (x_0, y_0) on the graph of f—i.e. (y_0, x_0) on the graph of g. The slope of the curve of f at this point is the reciprocal of the slope of the curve of g at the associated point.

Example

The function $f(x) = x^7 + x + 1$ has an inverse function g. Determine $g'(3)$.

we'll use $g'(y_0) = \dfrac{1}{f'(x_0)}$ where $f(x_0) = y_0$

i.e. $g(y_0) = x_0$

If $g'(3)$ is $g'(y_0)$, we need to find $f'(x_0)$.

For this, we need to know what x_0 is.

we'll find x_0 by educated guessing.

$y_0 = 3$, we need $f(x_0) = 3$

$$f(x_0) = x_0^7 + x_0 + 1 = 3$$

$x_0 = 1 \qquad f(x) = x^7 + x + 1$

$$\Rightarrow f'(x) = 7x^6 + 1$$

$$\Rightarrow f'(1) = 7(1)^6 + 1 = 8$$

Hence $\quad g'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{8}$

Inverse Trigonometric Functions

Recall the definitions of the inverse trigonometric functions.

$$y = \sin^{-1} x \quad \Longleftrightarrow \quad x = \sin y, \quad -1 \le x \le 1, \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$y = \cos^{-1} x \quad \Longleftrightarrow \quad x = \cos y, \quad -1 \le x \le 1, \quad 0 \le y \le \pi$$

$$y = \tan^{-1} x \quad \Longleftrightarrow \quad x = \tan y, \quad -\infty < x < \infty, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Inverse Trigonometric Functions

There are different conventions used for the ranges of the remaining functions. Sullivan and Miranda use

$$y = \cot^{-1} x \quad \Longleftrightarrow \quad x = \cot y, \quad -\infty < x < \infty, \quad 0 < y < \pi$$

$$y = \csc^{-1} x \quad \Longleftrightarrow \quad x = \csc y, \quad |x| \geq 1, \quad y \in \left(-\pi, -\frac{\pi}{2}\right] \cup \left(0, \frac{\pi}{2}\right]$$

$$y = \sec^{-1} x \quad \Longleftrightarrow \quad x = \sec y, \quad |x| \geq 1, \quad y \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$$

Derivative of the Inverse Sine

Use implicit differentiation to find $\frac{d}{dx}\sin^{-1}x$, and determine the interval over which $y=\sin^{-1}x$ is differentiable.

$y=\sin^{-1}x \implies x=\sin y$ where $-\pi/2 \le y \le \frac{\pi}{2}$

Take $\frac{d}{dx}$ of the relation $x=\sin y$.

$$\frac{d}{dx}x=\frac{d}{dx}\sin y$$

$$1=\cos y\cdot\frac{dy}{dx}$$

If $\cos y\neq 0$, we can divide. This requires

$y\neq\frac{\pi}{2}$ and $y\neq\frac{-\pi}{2}$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

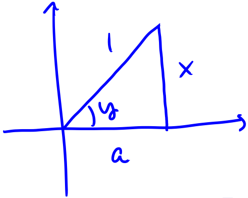
* Note: it's a reciprocal of $\frac{d}{dx}\sin x$

We want to know what $\frac{1}{\cos y}$ is in terms of x.

$y = \sin^{-1} x$ so $\cos y = \cos\left(\sin^{-1} x\right)$

$\sin^{-1} x$ is the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x.

$$\frac{\text{opp}}{\text{hyp}} = x = \frac{x}{1}$$



By the Pyth. Thm $a^2 + x^2 = 1^2$

$$a^2 = 1 - x^2 \Rightarrow a = \sqrt{1-x^2}$$

$$\cos y = \frac{\text{adj}}{\text{hyp}} = \frac{a}{1} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

$$\boxed{\text{So finally, } \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}}$$

Examples

Evaluate each derivative

(a) $\frac{d}{dx}\sin^{-1}(e^x)$ $= \frac{1}{\sqrt{1-(e^x)^2}} \cdot e^x$

outside $\sin^{-1} u$

inside $u = e^x$

$$= \frac{e^x}{\sqrt{1-e^{2x}}}$$

(b) $\frac{d}{dx}\left(\sin^{-1} x\right)^3$

outside u^3

$$= 3\left(\sin^{-1} x\right)^2 \cdot \frac{1}{\sqrt{1-x^2}}$$

inside $u = \sin^{-1} x$

$$= \frac{3\left(\sin^{-1} x\right)^2}{\sqrt{1-x^2}}$$

Derivative of the Inverse Tangent

Theorem: If $f(x) = \tan^{-1} x$, then f is differentiable for all real x and

$$f'(x) = \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}.$$

Questions

Find $\frac{dy}{dx}$ where $y = \tan^{-1} e^x$.

(a) $$\frac{dy}{dx} = \frac{e^x}{1 + e^{2x}}$$

(b) $$\frac{dy}{dx} = \frac{e^x}{1 + x^2}$$

(c) $$\frac{dy}{dx} = e^x \tan^{-1}$$

(d) $$\frac{dy}{dx} = \frac{1}{1 + e^{2x}}$$

Derivative of the Inverse Secant

Theorem: If $f(x) = \sec^{-1} x$, then f is differentiable for all $|x| > 1$ and

$$f'(x) = \frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}.$$

Examples

Evaluate

(a) $\dfrac{d}{dx}\sec^{-1}(x^2) = \dfrac{1}{x^2\sqrt{(x^2)^2-1}} \cdot (2x) = \dfrac{2x}{x^2\sqrt{x^4-1}}$

$= \dfrac{2}{x\sqrt{x^4-1}}$

(b) $\dfrac{d}{dx}\tan^{-1}(\sec x) = \dfrac{1}{1+(\sec x)^2} \cdot \sec x \tan x = \dfrac{\sec x \tan x}{1+\sec^2 x}$

The Remaining Inverse Functions

Due to the trigonometric cofunction identities, it can be shown that

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

and

$$\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$$

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}, \qquad \frac{d}{dx}\csc^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$$

Section 3.3: Derivatives of Logarithmic Functions

Recall: If $a > 0$ and $a \neq 1$, we denote the **base** a **logarithm** of x by

$$\log_a x$$

This is the inverse function of the (one to one) function $y = a^x$. So we can define $\log_a x$ by the statement

$$y = \log_a x \quad \text{if and only if} \quad x = a^y.$$

Our present goal is to use our knowledge of the derivative of an exponential function, along with the chain rule, to come up with a derivative rule for logarithmic functions.

Properties of Logarithms

We recall several useful properties of logarithms.

Let a, b, x, y be positive real numbers with $a \neq 1$ and $b \neq 1$, and let r be any real number.

- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
- $\log_a(x^r) = r\log_a(x)$
- $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$ (*the change of base formula*)
- $\log_a(1) = 0$

Question

(1) In the expression $\ln(x)$, what is the base?

(a) 10

(b) 1

(c) e

Question

(2) Which of the following expressions is equivalent to

$$\log_2\left(x^3\sqrt{y^2-1}\right) = \log_2\left(x^3 (y^2-1)^{1/2}\right)$$

$$= \log_2(x^3) + \log_2(y^2-1)^{1/2}$$

$$= 3\log_2 x + \frac{1}{2}\log_2(y^2-1)$$

(a) $\log_2(x^3) - \frac{1}{2}\log_2(y^2-1)$

(b) $\frac{3}{2}\log_2(x(y^2-1))$

(c) $3\log_2(x) + \frac{1}{2}\log_2(y^2-1)$

(d) $3\log_2(x) + \frac{1}{2}\log_2(y^2) - \frac{1}{2}\log_2(1)$

Properties of Logarithms

Additional properties that are useful.

- $f(x) = \log_a(x)$, has domain $(0, \infty)$ and range $(-\infty, \infty)$.

- For $a > 1$, *

$$\lim_{x\to 0^+} \log_a(x) = -\infty \quad \text{and} \quad \lim_{x\to \infty} \log_a(x) = \infty$$

- For $0 < a < 1$,

$$\lim_{x\to 0^+} \log_a(x) = \infty \quad \text{and} \quad \lim_{x\to \infty} \log_a(x) = -\infty$$

In advanced mathematics (and in light of the change of base formula), we usually restrict our attention to the natural log.

Graphs of Logarithms:Logarithms are continuous on $(0, \infty)$.

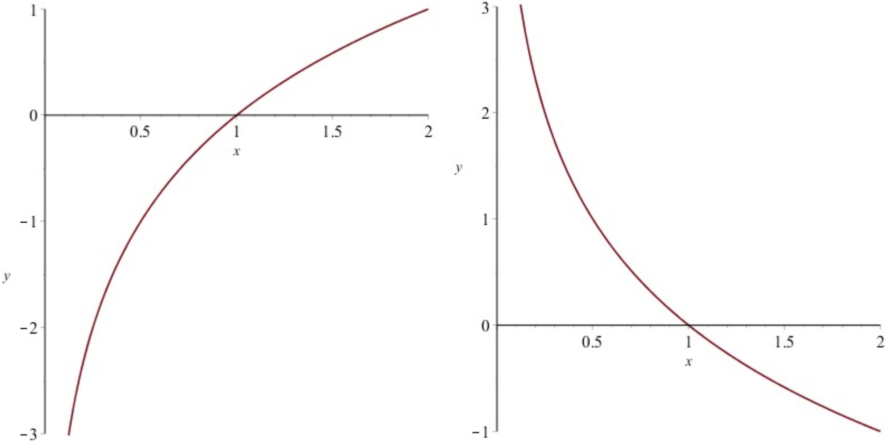
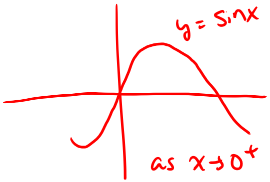


Figure: Plots of functions of the type $f(x) = \log_a(x)$. The value of $a > 1$ on the left, and $0 < a < 1$ on the right.

Examples

Evaluate each limit.

(a) $\displaystyle\lim_{x\to 0^+} \ln(\sin(x))$ $= -\infty$

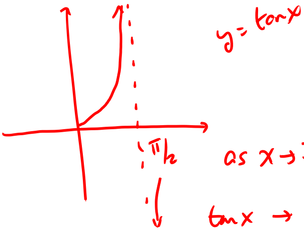


as $x \to 0^+$

$\sin x \to 0^+$

Base $e > 1$

(b) $\displaystyle\lim_{x\to \frac{\pi}{2}^-} \ln(\tan(x))$ $= \infty$



as $x \to \frac{\pi}{2}^-$

$\tan x \to \infty$

Question

Evaluate the limit $\lim_{x \to \infty} \ln\left(\frac{1}{x^2}\right)$

(a) $-\infty$

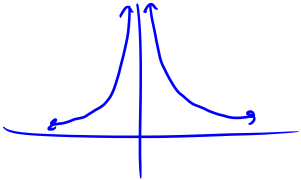
(b) 0

(c) ∞

(d) The limit doesn't exist.

$\lim_{x \to \infty} \frac{1}{x^2} = 0$

as $x \to \infty$, $\frac{1}{x^2} \to 0^+$



Logarithms are Differentiable on Their Domain

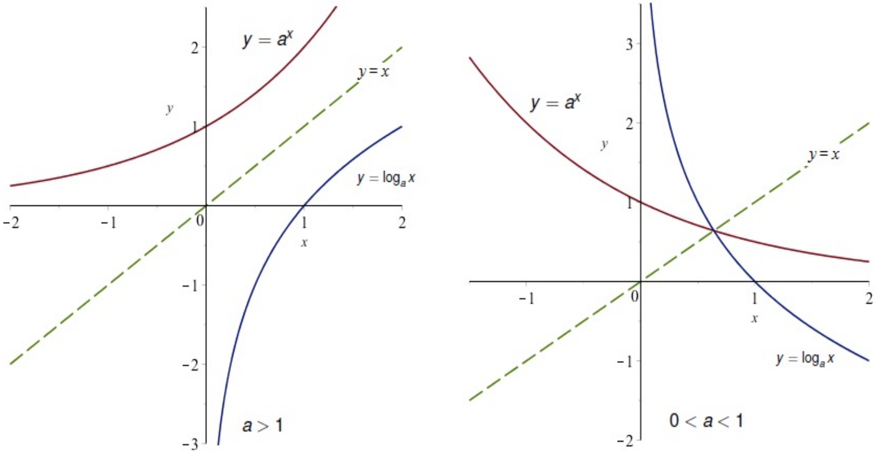


Figure: Recall $f(x) = a^x$ is differentiable on $(-\infty, \infty)$. The graph of $\log_a(x)$ is a reflection of the graph of a^x in the line $y = x$. So $f(x) = \log_a(x)$ is differentiable on $(0, \infty)$.

The Derivative of $y = \log_a(x)$

To find a derivative rule for $y = \log_a(x)$, we use the chain rule.

Let $y = \log_a(x)$, then $x = a^y$.

* Recall $\frac{d}{dx} a^x = a^x \ln a$

$$\frac{d}{dx} x = \frac{d}{dx} a^y$$

$$1 = a^y \ln a \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{a^y \ln a} \quad \text{but} \quad a^y = x \quad \text{so}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{x \ln a}}$$

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$

Examples: Evaluate each derivative.

(a) $\frac{d}{dx}\log_3(x)$ $= \frac{1}{x \ln 3}$

(b) $\frac{d}{d\theta}\log_{\frac{1}{2}}(\theta)$ $= \frac{1}{\theta \ln \frac{1}{2}}$

Question

True or False The derivative of the natural log

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}\ln x = \frac{d}{dx}\log_e x = \frac{1}{x\ln e} = \frac{1}{x\cdot 1} = \frac{1}{x}$$

The function $\ln|x|$

Show that if $x < 0$, then $\frac{d}{dx}\ln(-x) = \frac{1}{x}$.

Inside $\quad u = -x \quad$ so $\quad \frac{du}{dx} = -1$

outside $\quad \ln u, \quad$ so $\quad \frac{d}{du}\ln u = \frac{1}{u}$

so $\quad \frac{d}{dx}\ln(-x) = \frac{1}{-x} \cdot (-1) = \frac{-1}{-x} = \frac{1}{x}$

The function $\ln|x|$

Recall that $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$. We have the more general derivative rule

$$\frac{d}{dx}\ln|x| = \frac{1}{x}.$$

Differentiating Functions Involving Logs

We can combine our new rule with our existing derivative rules.

Chain Rule: Let u be a differentiable function. Then

$$\frac{d}{dx}\log_a|u| = \frac{1}{u\ln(a)}\frac{du}{dx} = \frac{u'(x)}{u(x)\ln(a)}.$$

In particular

$$\frac{d}{dx}\ln|u| = \frac{1}{u}\frac{du}{dx} = \frac{u'(x)}{u(x)}.$$

Examples

$$\frac{d}{dx} \ln|f(x)| = \frac{f'(x)}{f(x)}$$

Evaluate each derivative.

(a) $\dfrac{d}{dx} \ln|\tan x| = \dfrac{\sec^2 x}{\tan x}$

(b) $\dfrac{d}{dt} \log_2(3t^4+2t+7) = \dfrac{12t^3+2}{(3t^4+2t+7)\ln 2}$

Example

Determine $\frac{dy}{dx}$ if $\quad x \ln y + y \ln x = 10.$

$$\frac{d}{dx}\left(x \ln y + y \ln x\right) = \frac{d}{dx} 10$$

products

$$1 \cdot \ln y + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + \frac{dy}{dx} \ln x + y \cdot \frac{1}{x} = 0$$

$$\ln y + \frac{x}{y}\frac{dy}{dx} + (\ln x)\frac{dy}{dx} + \frac{y}{x} = 0$$

$$\frac{x}{y}\frac{dy}{dx} + (\ln x)\frac{dy}{dx} = -\ln y - \frac{y}{x}$$

we can clear fractions by multiplying by xy

$$xy\left(\frac{x}{y}\frac{dy}{dx} + (\ln x)\frac{dy}{dx}\right) = xy\left(-\ln y - \frac{y}{x}\right)$$

$$x^2\frac{dy}{dx} + xy(\ln x)\frac{dy}{dx} = -xy\ln y - y^2$$

$$(x^2 + xy\ln x)\frac{dy}{dx} = -xy\ln y - y^2$$

$$\boxed{\frac{dy}{dx} = \frac{-xy\ln y - y^2}{x^2 + xy\ln x}}$$

Questions

Find y' if $y = x(\ln x)^2$.

$$y' = 1 \cdot (\ln x)^2 + x\left(2(\ln x) \cdot \frac{1}{x}\right)$$

$$= (\ln x)^2 + 2\ln x$$

(a) $y' = \frac{2\ln x}{x}$

(b) $y' = 2\ln x + 2$

(c) $y' = (\ln x)^2 + 2\ln x$

(d) $y' = \ln(x^2) + 2$

Questions

Use implicit differentiation to find $\frac{dy}{dx}$ if $\quad y^2 \ln x = x + y$.

(a) $\dfrac{dy}{dx} = \dfrac{x - y^2}{2xy \ln x - x}$

(b) $\dfrac{dy}{dx} = \dfrac{1}{2y \ln x - 1}$

(c) $\dfrac{dy}{dx} = y^2 \ln x - 1$

(c) $\dfrac{dy}{dx} = \dfrac{x}{2y - x}$

Using Properties of Logs

Properties of logarithms can be used to simplify expressions characterized by products, quotients and powers.

Illustrative Example: Evaluate $\displaystyle \frac{d}{dx} \ln\left(\frac{x^2\cos(2x)}{\sqrt[3]{x^2+x}}\right)$

we do know $\displaystyle \frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$

we'll use log properties first, then take derivatives.

$$\ln\left(\frac{x^2\cos(2x)}{\sqrt[3]{x^2+x}}\right) = \ln\left(x^2\cos(2x)\right) - \ln\sqrt[3]{x^2+x}$$

$$= \ln x^2 + \ln(\cos(2x)) - \ln (x^2+x)^{1/3}$$

$$= 2\ln x + \ln(\cos(2x)) - \frac{1}{3}\ln(x^2+x)$$

$$\frac{d}{dx}\ln\left(\frac{x^2\cos(2x)}{\sqrt[3]{x^2+x}}\right) = \frac{d}{dx}\left(2\ln x + \ln((\cos(2x)) - \frac{1}{3}\ln(x^2+x)\right)$$

$$= 2\cdot\frac{1}{x} + \frac{-\sin(2x)\cdot 2}{\cos(2x)} - \frac{1}{3}\,\frac{2x+1}{x^2+x}$$

$$= \frac{2}{x} - 2\,\frac{\sin(2x)}{\cos(2x)} - \frac{1}{3}\,\frac{2x+1}{x^2+x}$$

$$\frac{d}{dx} \ln\left(\frac{x^2 \cos(2x)}{\sqrt[3]{x^2+x}}\right) = \frac{2}{x} - 2\tan(2x) - \frac{1}{3}\,\frac{2x+1}{x^2+x}$$

Question

Use properties of logs to expand completely $\ln\left(\dfrac{(x+1)(x+3)^3}{\sqrt{x}\sin x}\right)$.

(a) $\ln(x+1)+\ln(x+3)^3-\ln\sqrt{x}+\ln\sin x$

(b) $\ln(x+1)+3\ln(x+3)-\dfrac{1}{2}\ln x+\ln\sin x$

(c) $\ln(x+1)+3\ln(x+3)-\dfrac{1}{2}\ln x-\ln\sin x$

(d) $\ln(x+1)+\ln(x+3)^3-\dfrac{1}{2}\ln x-\ln\sin-\ln x$

Logarithmic Differentiation

Expressions consisting of complicated powers, products, and quotients may be differentiated by introducing a log.

Evaluate $\quad \dfrac{d}{dx}\left(\dfrac{x^2\sqrt{x+1}}{\cos^4(3x)}\right)$

There's no log here, but we'll introduce one.

Let $y = \dfrac{x^2\sqrt{x+1}}{\cos^4(3x)}$. Then $\ln y = \ln\left(\dfrac{x^2\sqrt{x+1}}{\cos^4(3x)}\right)$

Using log properties

$$\ln y = \ln\left(\frac{x^2\sqrt{x+1}}{\cos^4(3x)}\right) = \ln\left(x^2(x+1)^{1/2}\right) - \ln\left(\cos^4(3x)\right)$$

$$= \ln x^2 + \ln (x+1)^{1/2} - \ln (\cos(3x))^4$$

$$= 2\ln x + \tfrac{1}{2}\ln(x+1) - 4\ln \cos(3x)$$

we have

$$\ln y = 2\ln x + \tfrac{1}{2}\ln(x+1) - 4\ln \cos(3x)$$

we'll take $\frac{d}{dx}$ of both sides. we'll have to use the fact that

$$\frac{d}{dx}\ln y = \frac{1}{y}\frac{dy}{dx}$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(2\ln x + \frac{1}{2}\ln(x+1) - 4\ln\cos(3x)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = 2\cdot\frac{1}{x} + \frac{1}{2}\frac{1}{x+1} - 4\,\frac{-\sin(3x)\cdot 3}{\cos(3x)}$$

$$= \frac{2}{x} + \frac{1}{2}\frac{1}{x+1} + 12\tan(3x)$$

mult. by y

$$\frac{dy}{dx} = y\left(\frac{2}{x} + \frac{1}{2}\frac{1}{x+1} + 12\cdot\tan(3x)\right)$$

Sub in $y = \dfrac{x^2\sqrt{x+1}}{\cos^4(3x)}$

$$\boxed{\frac{dy}{dx} = \frac{x^2\sqrt{x+1}}{\cos^4(3x)}\left(\frac{2}{x} + \frac{1}{2}\frac{1}{x+1} + 12\tan(3x)\right)}$$

Logarithmic Differentiation

If the differentiable function $y = f(x)$ consists of complicated products, quotients, and powers:

(i) Take the logarithm of both sides, i.e. $\ln(y) = \ln(f(x))$. Then use properties of logs to express $\ln(f(x))$ as a sum/difference of simpler terms.

(ii) Take the derivative of each side, and use the fact that $\frac{d}{dx}\ln(y) = \frac{\frac{dy}{dx}}{y}$.

(iii) Solve for $\frac{dy}{dx}$ (i.e. multiply through by y), and replace y with $f(x)$ to express the derivative explicitly as a function of x.

When **only** Log. Differentiation can be used:

Find $\frac{dy}{dx}$ if $y = x^{\sin x}$.

power function x^n
variable base, constant power

exponential function e^x
Constant base, variable power

variable power and variable base!

There is no derivative formula for y.

The only way to find $\frac{dy}{dx}$ is to use log. differentiation.

$$\ln y = \ln x^{\sin x} = \underbrace{\sin x \ln x}_{\text{now we have a product!}}$$

$$\frac{d}{dx} \ln y = \frac{d}{dx} \left(\sin x \ln x \right)$$

$$\frac{1}{y} \frac{dy}{dx} = \cos x \ln x + \sin x \left(\frac{1}{x} \right)$$

$$\frac{dy}{dx} = y \left(\cos x \ln x + \frac{\sin x}{x} \right)$$

$$\frac{dy}{dx} = x^{\sin x} \left(\cos x \ln x + \frac{\sin x}{x} \right)$$

y