June 19 Math 2254 sec 001 Summer 2015

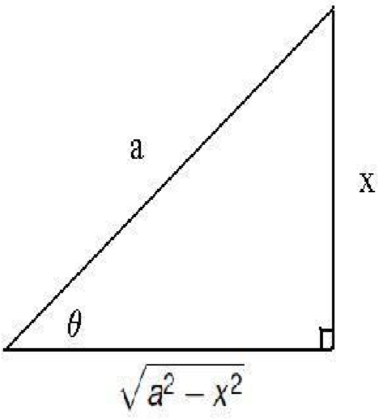
Section 7.3 Trigonometric Substitution

Substitution for the form $\sqrt{a^2 - x^2}$

$$x = a\sin\theta$$

$$dx = a\cos\theta\, d\theta$$

$$\sqrt{a^2 - x^2} = a\cos\theta$$

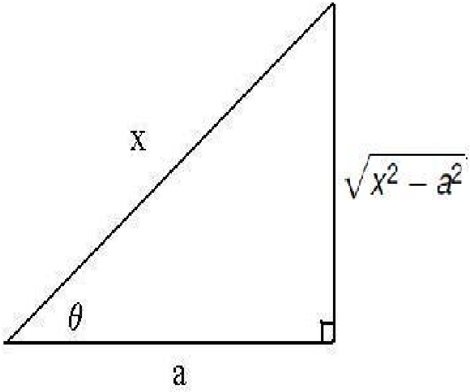


Substitution for the form $\sqrt{x^2 - a^2}$

$$x = a\sec\theta$$

$$dx = a\sec\theta\tan\theta\, d\theta$$

$$\sqrt{x^2 - a^2} = a\tan\theta$$

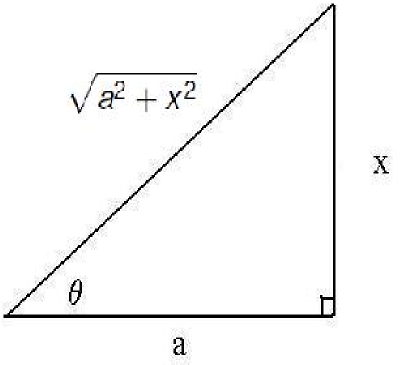


Substitution for the form $\sqrt{a^2+x^2}$

$$x = a\tan\theta$$

$$dx = a\sec^2\theta\, d\theta$$

$$\sqrt{x^2+a^2} = a\sec\theta$$



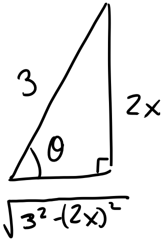
We'll assume that θ is in an appropriate interval–e.g. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for the substitution $x = a\tan\theta$

Evaluate The Integral

(f) $\displaystyle\int_0^1 \sqrt{9-4x^2}\,dx \quad = \int_0^1 \sqrt{3^2-(2x)^2}\;dx$

Let's evaluate the indefinite integral

$$\int \sqrt{3^2-(2x)^2}\;dx$$



$*\;\theta = \sin^{-1}\left(\frac{2x}{3}\right) \leftrightarrow$ $\qquad \sin\theta = \dfrac{2x}{3} = \dfrac{2}{3}x$

$$x = \frac{3}{2}\sin\theta$$

$$dx = \frac{3}{2}\cos\theta \, d\theta \qquad \cos\theta = \frac{\sqrt{9-4x^2}}{3}$$

$$\sqrt{9-4x^2} = 3\cos\theta$$

$$\int \sqrt{9-4x^2}\, dx = \int 3\cos\theta \cdot \frac{3}{2}\cos\theta \, d\theta$$

$$* \ \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{9}{2}\int \cos^2\theta \, d\theta$$

$$= \frac{9}{2}\int \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{9}{4}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{9}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{9}{4}\left(\theta + \frac{1}{2}\cdot 2\sin\theta\cos\theta\right) + C$$

$$= \frac{9}{4}\left(\sin^{-1}\left(\frac{2x}{3}\right) + \frac{2x}{3}\,\frac{\sqrt{9-4x^2}}{3}\right) + C$$

$$\int_0^1 \sqrt{9-4x^2}\,dx = \frac{9}{4}\sin^{-1}\left(\frac{2x}{3}\right) + \frac{1}{2}x\sqrt{9-4x^2}\ \Bigg|_0^1$$

$$= \frac{9}{4}\sin^{-1}\left(\frac{2}{3}\right) + \frac{1}{2}\sqrt{9-4} - \left(\frac{9}{4}\sin^{-1}(0) + 0\right)$$

$$= \frac{9}{4}\sin^{-1}\left(\frac{2}{3}\right) + \frac{1}{2}\sqrt{5}$$

Evaluate The Integral

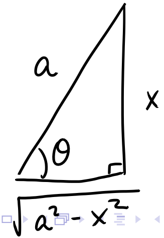
Derive the three formulas

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \quad \text{and}$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}}$$



$$\sin\theta = \frac{x}{a} \Rightarrow x = a\sin\theta$$

$$dx = a\cos\theta\, d\theta$$

$$\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$$

$$\sqrt{a^2-x^2} = a\cos\theta$$

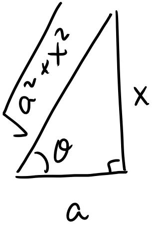
$$\int \frac{dx}{\sqrt{a^2-x^2}}$$

$$= \int \frac{a\cos\theta\, d\theta}{a\cos\theta}$$

$$= \int d\theta = \theta + C$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{a^2+x^2} = \int \frac{dx}{\left(\sqrt{a^2+x^2}\right)^2}$$



$$= \int \frac{a\sec^2\theta\, d\theta}{(a\sec\theta)^2}$$

$$\tan\theta = \frac{x}{a} \Rightarrow x = a\tan\theta$$

$$= \int \frac{a\sec^2\theta}{a^2\sec^2\theta}\, d\theta$$

$$dx = a\sec^2\theta\, d\theta$$

$$\sec\theta = \frac{\sqrt{a^2+x^2}}{a}$$

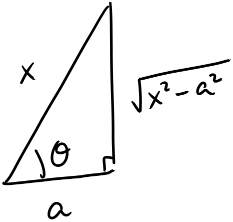
$$= \int \frac{1}{a}\, d\theta$$

$$\sqrt{a^2+x^2} = a\sec\theta$$

$$= \frac{1}{a}\theta + C$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}}$$



$$\int \frac{dx}{x\sqrt{x^2-a^2}}$$

$$\sec\theta = \frac{x}{a} \Rightarrow x = a\sec\theta$$

$$dx = a\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{a\sec\theta\tan\theta\,d\theta}{a\sec\theta\;a\tan\theta}$$

$$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$$

$$\sqrt{x^2-a^2} = a\tan\theta$$

$$= \int \frac{1}{a}\,d\theta$$

$$= \frac{1}{a}\theta + C \quad = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$$