June 21 Math 1190 sec. 51 Summer 2017

Find $\frac{dy}{dx}$. $\quad y = \dfrac{x^3(4x-1)^5}{\sqrt[4]{x+5}}$

We'll use logarithmic differentiation. First, we'll take the ln of both sides.

$$\ln y = \ln\left(\frac{x^3(4x-1)^5}{\sqrt[4]{x+5}}\right)$$

Now use log properties

$$= \ln\left(x^3(4x-1)^5\right) - \ln\sqrt[4]{x+5}$$

$$= \ln x^3 + \ln (4x-1)^5 - \ln (x+5)^{\frac{1}{4}}$$

$$= 3\ln x + 5\ln(4x-1) - \frac{1}{4}\ln(x+5)$$

$$\ln y = 3\ln x + 5\ln(4x-1) - \frac{1}{4}\ln(x+5)$$

Take $\frac{d}{dx}$ of both sides

$$\frac{1}{y}\frac{dy}{dx} = 3\cdot\frac{1}{x} + 5\,\frac{4}{4x-1} - \frac{1}{4}\,\frac{1}{x+5}$$

$$\frac{dy}{dx} = y\left(\frac{3}{x} + \frac{20}{4x-1} - \frac{1}{4(x+5)}\right)$$

sub in y ↓

$$\boxed{\frac{dy}{dx} = \frac{x^3(4x-1)^5}{\sqrt[4]{x+5}}\left(\frac{3}{x} + \frac{20}{4x-1} - \frac{1}{4(x+5)}\right)}$$

When **only** Log. Differentiation can be used:

Find $\frac{dy}{dx}$ if $y = (\ln x)^{2x}$.

$$\ln y = \ln (\ln x)^{2x} = 2x \ln(\ln x)$$

$*$ Note $\frac{d}{dx} \ln(\ln x)$

$$= \frac{1}{\ln x} \cdot \frac{1}{x}$$

Letting $u = \ln x$ $\quad \frac{du}{dx} = \frac{1}{x}$

and $f(u) = \ln u$ $\quad f'(u) = \frac{1}{u}$

$$\frac{d}{dx} \ln y = \frac{d}{dx}\left(2x \ln(\ln x)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = 2\cdot\ln(\ln x) + 2x\cdot\frac{1}{\ln x}\cdot\frac{1}{x}$$

$$\frac{dy}{dx} = y\left(2\ln(\ln x) + \frac{2}{\ln x}\right)$$

$$\boxed{\frac{dy}{dx} = (\ln x)^{2x}\left(2\ln(\ln x) + \frac{2}{\ln x}\right)}$$

Question

A particle moves along the x-axis so that its position relative to the origin is $s(t) = \ln(t^2 + 1)$. The velocity v and acceleration a are

(a) $v = \dfrac{1}{t^2 + 1}$, and $a = \dfrac{-2t}{(t^2 + 1)^2}$

(b) $v = \dfrac{2}{t}$, and $a = \dfrac{-2}{t^2}$

(c) $v = \dfrac{2t}{t^2 + 1}$, and $a = \dfrac{2 - 2t^2}{(t^2 + 1)^2}$

(d) $v = \dfrac{2t}{t^2 + 1}$, and $a = \dfrac{2}{2t}$

Section 4.5: Indeterminate Forms & L'Hôpital's Rule

Consider the following three limit statements (all of which are true):

(a) $$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$$

(b) $$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

(c) $$\lim_{x\to 3} \frac{x^2-9}{(x-3)^2} \quad \text{doesn't exist}$$

Note: Each of these three limits involve both numerator and denominator going to zero—giving the form $\frac{0}{0}$. In the top two, the limit exists, but the limits are different. In the third, the limit doesn't exist.

Indeterminate Forms

$0/0$ is called an **Indeterminate form**.

Other indeterminate forms we'll encounter include

$$\frac{\pm\infty}{\pm\infty}, \quad \infty - \infty, \quad 0\infty, \quad 1^{\infty}, \quad 0^{0}, \quad \text{and} \quad \infty^{0}.$$

Indeterminate forms are not defined (as number)

Question

(1) **True or False:** $\infty - \infty = 0$.

↑ indeterminate form

(2) **True or False:** The form $\frac{1}{0}$ is indeterminate.

"$\frac{1}{0}$" indicates a non existing infinite limit

(3) **True or False:** $\frac{0}{1} = 0$.

Theorem: l'Hospital's Rule (part 1)

Suppose f and g are differentiable on an open interval I containing c (except possibly at c), and suppose $g'(x) \neq 0$ on I. If

$$\lim_{x\to c} f(x) = 0 \quad \text{and} \quad \lim_{x\to c} g(x) = 0$$

then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is ∞ or $-\infty$).

If $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)}$ looks like "$\dfrac{0}{0}$", we can try to find $\lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)}$

Note $\dfrac{f'(x)}{g'(x)}$ is not $\dfrac{d}{dx}\dfrac{f(x)}{g(x)}$, the derivatives of f and g are separate.

Evaluate each limit if possible

(a) $\lim_{x\to 1} \frac{\ln x}{x-1}$ $= " \frac{0}{0} "$

Note: $\ln 1 = 0$ and $1-1=0$

so $\lim_{x\to 1} \ln x = 0$ and $\lim_{x\to 1} (x-1) = 0$

apply l'H

l'Hospital's Rule applies

$$= \lim_{x\to 1} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx}(x-1)}$$

$$= \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \frac{\frac{1}{1}}{1} = 1$$

Theorem: l'Hospital's Rule (part 2)

Suppose f and g are differentiable on an open interval I containing c (except possibly at c), and suppose $g'(x) \neq 0$ on I. If

$$\lim_{x \to c} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to c} g(x) = \pm\infty$$

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is ∞ or $-\infty$).

If $\lim_{x \to c} \frac{f(x)}{g(x)}$ looks like $\frac{\pm\infty}{\pm\infty}$, try taking

$$\lim_{x \to c} \frac{f'(x)}{g'(x)}$$

(b) $\lim_{x\to\infty} xe^{-x}$ = "$\infty \cdot 0$"

$\lim_{x\to\infty} e^{-x} = 0$

"$\infty \cdot 0$" is an indeterminate form. But for l'H rule, we need $\frac{0}{0}$ or $\frac{\infty}{\infty}$. We can turn the product into a quotient.

$$f(x)g(x) = \frac{f(x)}{1/g(x)} \quad \text{or} \quad f(x)g(x) = \frac{g(x)}{1/f(x)}$$

We can write $xe^{-x} = \dfrac{x}{\frac{1}{e^{-x}}} = \dfrac{x}{e^x}$ or $xe^{-x} = \dfrac{e^{-x}}{1/x}$

we'll use this one!

$$\lim_{x\to\infty} x e^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = \text{"}\frac{\infty}{\infty}\text{"}$$ l'H rule applies

use l'H

$$= \lim_{x\to\infty} \frac{\frac{d}{dx} x}{\frac{d}{dx} e^x}$$

$$= \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$$

(c) $\lim_{x \to 0} \frac{\cos x - 1}{x^2} = \text{"} \frac{0}{0} \text{"}$

Recall $\cos 0 = 1$

Use l'H

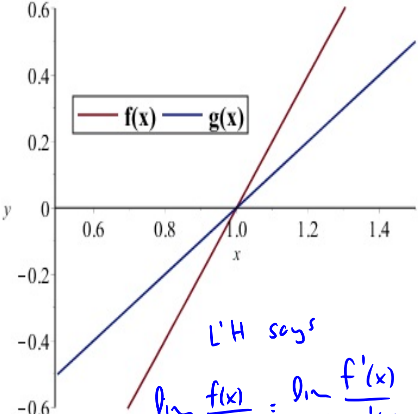
$$= \lim_{x \to 0} \frac{\frac{d}{dx}(\cos x - 1)}{\frac{d}{dx} x^2}$$

$$= \lim_{x \to 0} \frac{-\sin x}{2x} = \text{"} \frac{0}{0} \text{"}$$

use l'H rule again

$$= \lim_{x \to 0} \frac{-\cos x}{2} = \frac{-\cos 0}{2} = \frac{-1}{2}$$

Question



L'H says

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{f'(x)}{g'(x)}$$

$y = f(x)$ and $y = g(x)$ close to $x = 1$ are plotted on the same set of axes. Note that

$$\lim_{x\to 1} f(x) = 0 \quad \text{and} \quad \lim_{x\to 1} g(x) = 0$$

From the graph, only one of the following limit statements could be true. Which one?

(a) $\lim_{x\to 1} \frac{f(x)}{g(x)} = 0$

(b) $\lim_{x\to 1} \frac{f(x)}{g(x)} = 2$

(c) $\lim_{x\to 1} \frac{f(x)}{g(x)} = -2$

l'Hospital's Rule is not a "Fix-all"

Evaluate $\displaystyle \lim_{x\to 0^+} \frac{\cot x}{\csc x} = \text{"}\frac{\infty}{\infty}\text{"}$

$\displaystyle \lim_{x\to 0^+} \csc x = \infty$ and

$\displaystyle \lim_{x\to 0^+} \cot x = \infty$

use l'H rule $\displaystyle = \lim_{x\to 0^+} \frac{-\csc^2 x}{-\csc x \cot x}$

$\displaystyle = \lim_{x\to 0^+} \frac{\csc x}{\cot x} = \text{"}\frac{\infty}{\infty}\text{"}$

use l'H rule again $\displaystyle = \lim_{x\to 0^+} \frac{-\csc x \cot x}{-\csc^2 x} = \lim_{x\to 0^+} \frac{\cot x}{\csc x}$

We can use trig IDs instead.

$$\cot x = \frac{\cos x}{\sin x} \quad \text{and} \quad \csc x = \frac{1}{\sin x} \quad \text{so}$$

$$\frac{\cot x}{\csc x} = \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}} = \frac{\cos x}{\sin x} \cdot \frac{\sin x}{1} = \cos x$$

So $$\lim_{x\to 0^+} \frac{\cot x}{\csc x} = \lim_{x\to 0^+} \cos x = \underline{1}$$

Don't apply it if it doesn't apply!

$$\lim_{x \to 2} \frac{x+4}{x^2-3} = \frac{6}{1} = 6$$

BUT

$$\lim_{x \to 2} \frac{\frac{d}{dx}(x+4)}{\frac{d}{dx}(x^2-3)} \quad = \quad \lim_{x \to 2} \frac{1}{2x} \quad = \quad \frac{1}{4}$$

Remarks:

- l'Hopital's rule **only applies directly** to the forms $0/0$, or $(\pm\infty)/(\pm\infty)$.
- Multiple applications may be needed, or it may not result in a solution.
- It can be applied indirectly to the form $0 \cdot \infty$ by turning the product into a quotient.
- Derivatives of numerator and denominator are taken **separately**–this is NOT a *quotient rule* application.
- Applying it where it doesn't belong likely produces nonsense!

Question

True or False: If $\lim_{x \to c} f(x)g(x)$ produces the indeterminate form

$$0 \cdot \infty$$

then we apply l'Hopital's rule by considering

$$\lim_{x \to c} f'(x) \cdot g'(x)$$

It's only for $\frac{0}{0}$ or $\frac{\infty}{\infty}$

The form $\infty - \infty$

Evaluate the limit if possible

$$\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$$

$$= \infty - \infty$$

$$\lim_{x\to 1^+} \frac{1}{\ln x} = \infty \quad \text{and} \quad \lim_{x\to 1^-} \frac{1}{\ln x} = -\infty$$

$$\text{and} \quad \lim_{x\to 1^+} \frac{1}{x-1} = \infty \quad \text{while}$$

$$\lim_{x\to 1^-} \frac{1}{x-1} = -\infty$$

Do some algebra

$$\frac{1}{\ln x} - \frac{1}{x-1} = \frac{x-1}{\ln x(x-1)} - \frac{\ln x}{\ln x(x-1)} = \frac{x-1-\ln x}{\ln x\,(x-1)}$$

$$\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1} \frac{x-1-\ln x}{\ln x\,(x-1)} = \text{“}\frac{0}{0}\text{”}$$

use l'H rule $= \lim_{x\to 1} \dfrac{1-\frac{1}{x}}{\frac{1}{x}(x-1)+\ln x}$ let's clear the fractions

$$= \lim_{x\to 1} \frac{1-\frac{1}{x}}{\frac{1}{x}(x-1)+\ln x} \cdot \frac{x}{x}$$

$$= \lim_{x\to 1} \frac{x-1}{x-1+x\ln x} = \text{"}\frac{0}{0}\text{"}$$

use l'H again $= \lim_{x\to 1} \dfrac{1}{1+1\cdot\ln x + x\cdot\frac{1}{x}} = \dfrac{1}{1+\ln 1+1\cdot\frac{1}{1}}$

$$= \frac{1}{1+1} = \frac{1}{2}$$

Indeterminate Forms 1^∞, 0^0, and ∞^0

Since the logarithm and exponential functions are continuous, and $\ln(x^r) = r\ln x$, we have

$$\lim_{x\to a} F(x) = \exp\left(\ln\left[\lim_{x\to a} F(x)\right]\right) = \exp\left(\lim_{x\to a}\ln F(x)\right)$$

provided this limit exists.

- we want $\lim_{x\to a} F(x)$ but its 1^∞ or 0^0 or ∞^0

- we take $\lim_{x\to a} \ln F(x)$. If $\lim_{x\to a} \ln F(x) = L$

- then $\lim_{x\to a} F(x) = e^L$

Use this property to show that

$$\lim_{x\to 0}(1+x)^{1/x} = e$$

Note $1+x \to 1$ as $x \to 0$

and $\frac{1}{x} \to \infty$ as $x \to 0$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \text{"} 1^{\infty} \text{"}$$

We'll take the limit of $\ln(1+x)^{\frac{1}{x}}$.

$$\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}} = \lim_{x\to 0}\frac{1}{x}\ln(1+x)$$

$$= \lim_{x\to 0}\frac{\ln(1+x)}{x} = \text{"}\frac{0}{0}\text{"}$$

use l'H rule $\quad = \lim_{x\to 0} \dfrac{\frac{1}{1+x}}{1} = \dfrac{\frac{1}{1+0}}{1} = 1$

so $\lim_{x\to 0} \ln(1+x)^{\frac{1}{x}} = 1$.

and $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e^1 = e$

Question

True or False: Since $1^n = 1$ for every integer n, we should conclude that the indeterminate form 1^∞ is equal to 1.

Question

The limit

$$\lim_{x\to\infty} x^{1/x} \quad \text{gives rise to the indeterminate form}$$

(a) $\frac{\infty}{\infty}$

(b) ∞^0

(c) 0^0

(d) 1^∞

Question

$$\text{Since} \quad \ln\left(x^{1/x}\right) = \frac{1}{x}\ln x = \frac{\ln x}{x}, \quad \text{evaluate} \quad \lim_{x\to\infty}\frac{\ln x}{x}$$

(use l'Hopital's rule as needed)

(a) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x} = 0$

(b) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x} = 1$

(c) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x} = \infty$

Question

$$\lim_{x\to\infty} x^{1/x} = e^{0} = 1$$

(a) 0

(b) 1

(c) ∞

Section 4.2: Maximum and Minimum Values; Critical Numbers

Definition: Let f be a function with domain D and let c be a number in D. Then $f(c)$ is

- **the absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D,

- **the absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D.

Note that if an absolute minimum occurs at c, then $f(c)$ is the **absolute minimum value** of f. Similarly, if an absolute maximum occurs at c, then $f(c)$ is the **absolute maximum value** of f.

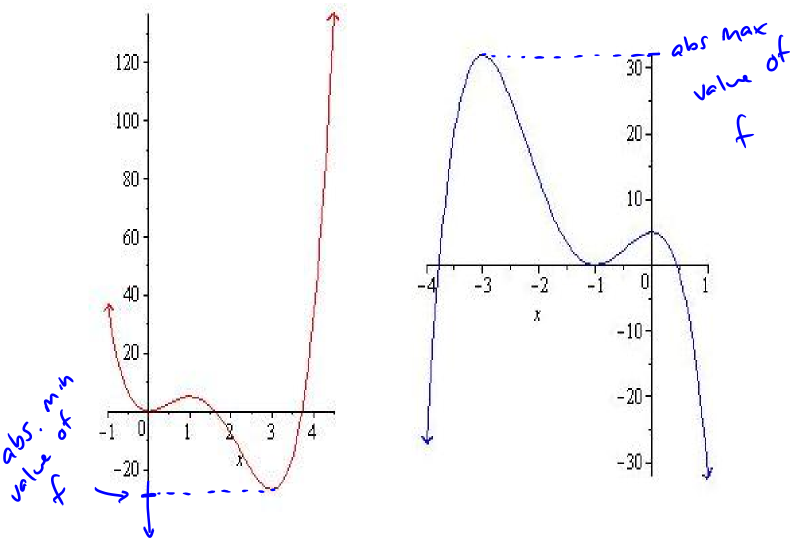


Figure: Graphically, an absolute minimum is the lowest point and an absolute maximum is the highest point.

Local Maximum and Minimum

Definition: Let f be a function with domain D and let c be a number in D. Then $f(c)$ is

- **a local minimum** value of f if $f(c) \le f(x)$ for x *near** c
- **a local maximum** value of f if $f(c) \ge f(x)$ for x *near* c.

More precisely, to say that x is *near* c means that there exists an open interval containing c such that for all x in this interval the respective inequality holds.

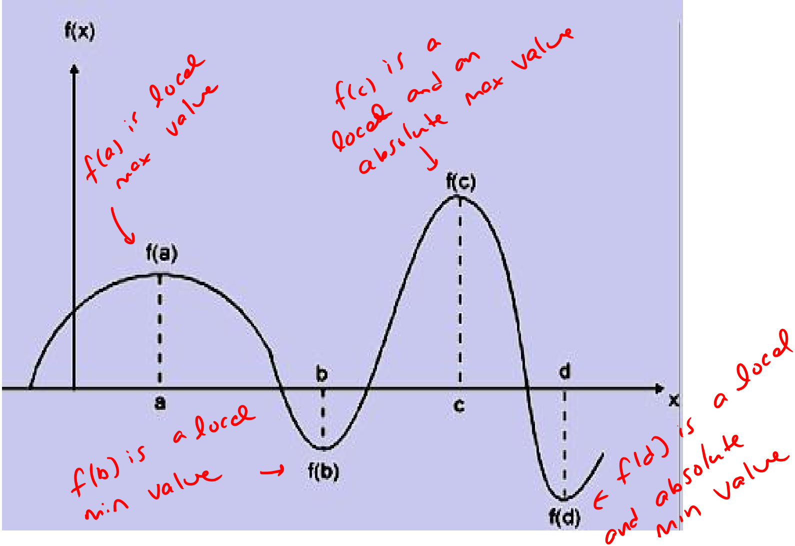


Figure: Graphically, local maxes and mins are *relative* high and low points.

Terminology

Maxima—-plural of maximum

Minima—-plural of minimum

Extremum——is either a maximum or a minimum

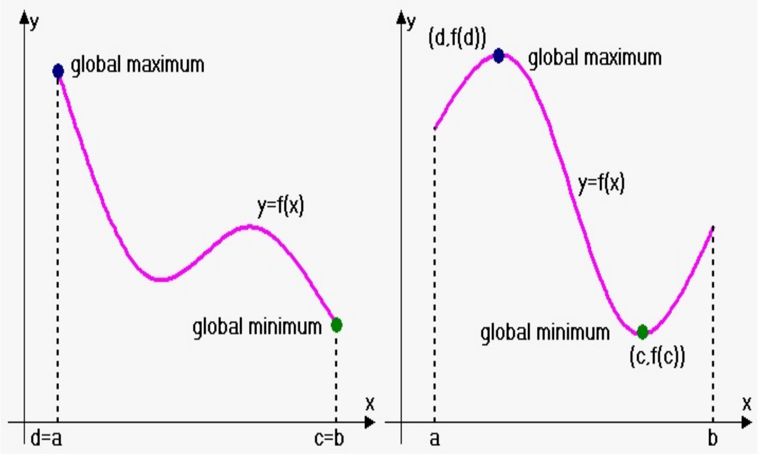
Extrema——plural of extremum

”**Global**” is another word for absolute.

”**Relative**” is another word for local.

Extreme Value Theorem

Suppose f is continuous on a closed interval $[a, b]$. Then f attains an absolute maximum value $f(d)$ and f attains an absolute minimum value $f(c)$ for some numbers c and d in $[a, b]$.



Fermat's Theorem

Note that the Extreme Value Theorem tells us that a continuous function is guaranteed to take an absolute maximum and absolute minimum on a closed interval. It does not provide a method for actually finding these values or where they occur. For that, the following theorem due to Fermat is helpful.

Theorem: If f has a local extremum at c and if $f'(c)$ exists, then

$$f'(c) = 0.$$

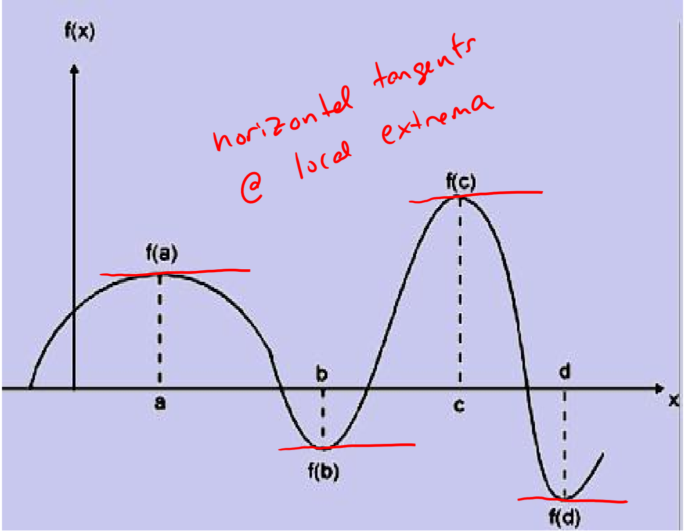


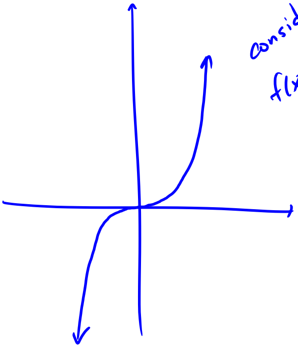
Figure: We note that at the local extrema, the tangent line would be horizontal.

Is the Converse of our Theorem True?

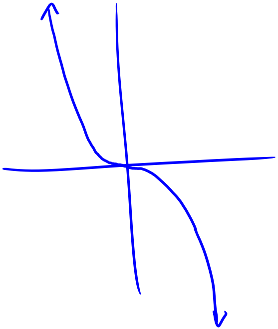
Suppose a function f satisfies $f'(0) = 0$. Can we conclude that $f(0)$ is a local maximum or local minimum?

No, there is another possibility:

consider $f(x) = x^3$



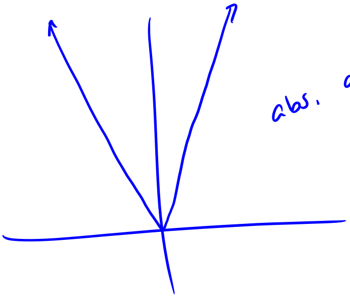
or $y = -x^3$



Does an extremum have to correspond to a horizontal tangent?

Could $f(c)$ be a local extremum but have $f'(c)$ not exist? Yes

Consider $f(x) = |x|$



abs. and local min @ $(0,0)$

But $f'(0)$ DNE

Critical Number

Definition: A **critical number** of a function f is a number c in its domain such that either

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

Theorem: If f has a local extremum at c, then c is a critical number of f.

Some authors call critical numbers *critical points*.

Example

Find all of the critical numbers of the function.

$$g(t) = t^{1/5}(12-t)$$

We need to know for which t-values $g'(t)=0$ and $g'(t)$ DNE.

$g(t) = 12t^{1/5} - t^{6/5}$

Find $g'(t)$: $g'(t) = 12\left(\frac{1}{5}t^{-4/5}\right) - \frac{6}{5}t^{1/5}$

$g'(t) = \dfrac{12}{5t^{4/5}} - \dfrac{6t^{1/5}}{5}$

write as one fraction

$= \dfrac{12}{5t^{4/5}} - \dfrac{6t^{1/5}}{5} \cdot \dfrac{t^{4/5}}{t^{4/5}}$

$$= \frac{12}{5t^{4/5}} - \frac{6t}{5t^{4/5}}$$

$$\Rightarrow \quad g'(t) = \frac{12-6t}{5t^{4/5}}$$

Recall : a fraction $=0$ if the numerator $=0$

a fraction is Und. if the denominator $=0$

$$g'(t) = 0 \quad \Rightarrow \quad 12-6t=0 \quad \Rightarrow \quad t = \frac{12}{6} = 2$$

$$g'(t) \text{ DNE} \quad \Rightarrow \quad 5t^{4/5} = 0 \quad \Rightarrow \quad t = 0$$

g has two critical numbers, 0 and 2.

Question

Find all of the critical numbers of the function.

$$f(x) = xe^x$$

$f'(x) = 1 \cdot e^x + x \cdot e^x$

$= e^x + xe^x = (1+x)e^x$

(a) -1 and 0

$f'(x)$ DNE never

(b) -1

$f'(x) = 0 \Rightarrow 0 = (1+x)e^x$

(c) -1 and e

$\Rightarrow 1+x=0$ or $e^x = 0$

$x = -1$ no soln

(d) There are none