June 22 Math 2254 sec 001 Summer 2015

Section 7.3 Trigonometric Substitution

Evaluate the integral using both options on the legs of the triangle.

$$\int \frac{x^2}{(16-x^2)^{3/2}}\,dx = \int \frac{x^2}{\left(\sqrt{16-x^2}\right)^3}\,dx$$

$$\int \frac{x^2}{\left(\sqrt{16-x^2}\right)^3}\,dx$$

$$= \int \frac{(4\sin\theta)^2\, 4\cos\theta\, d\theta}{(4\cos\theta)^3}$$

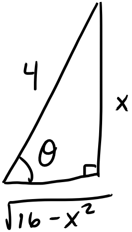
$\sin\theta = \frac{x}{4}$

$x = 4\sin\theta$

$dx = 4\cos\theta\, d\theta$

$\cos\theta = \frac{\sqrt{16-x^2}}{4}$

$\sqrt{16-x^2} = 4\cos\theta$



$$= \int \frac{4^2 \sin^2\theta \, 4\cos\theta}{4^3 \cos^3\theta} \, d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta} \, d\theta = \int \tan^2\theta \, d\theta$$

$\sin\theta = \frac{x}{4} \Rightarrow$

$\theta = \sin^{-1}\left(\frac{x}{4}\right)$

$$= \int (\sec^2\theta - 1) \, d\theta$$

$$= \tan\theta - \theta + C$$

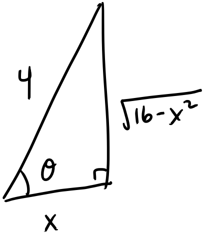
$$= \frac{x}{\sqrt{16 - x^2}} - \sin^{-1}\left(\frac{x}{4}\right) + C$$

Repeat:

$$\int \frac{x^2}{\left(\sqrt{16-x^2}\right)^3}\,dx$$

$$= \int \frac{(4\cos\theta)^2(-4\sin\theta)\,d\theta}{(4\sin\theta)^3}$$

$$= -\int \frac{4^2\cos^2\theta \cdot 4\sin\theta\,d\theta}{4^3\sin^3\theta}$$



$$\cos\theta = \frac{x}{4}$$

$$x = 4\cos\theta$$

$$dx = -4\sin\theta\,d\theta$$

$$\sin\theta = \frac{\sqrt{16-x^2}}{4}$$

$$\sqrt{16-x^2} = 4\sin\theta$$

$$= -\int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$\cos\theta = \frac{x}{4}$$

$$\theta = \cos^{-1}\left(\frac{x}{4}\right)$$

$$= -\int \cot^2\theta \, d\theta$$

$$= -\int (\csc^2\theta - 1)\, d\theta$$

$$= -(-\cot\theta - \theta) + K$$

$$= \cot\theta + \theta + K$$

$$= \frac{x}{\sqrt{16-x^2}} + \cos^{-1}\left(\frac{x}{4}\right) + K$$

$$= \frac{x}{\sqrt{16-x^2}} - \sin^{-1}\left(\frac{x}{4}\right) + C \quad \text{from before}$$

Recall $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} = -\frac{d}{dx}\cos^{-1}x$

since $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

Section 7.4: Substitution with Quadratics $(ax^2 + bx + c)$

Recall that we can complete the square on a quadratic ($a \neq 0$)

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \\ &= au^2 + r \end{aligned}$$

add $\left(\frac{b}{2a}\right)^2$ and subtract

where $u = x + \frac{b}{2a}$ and $r = c - \frac{b^2}{4a}$.

Examples

(a) $$\int \frac{dt}{t^2 - 6t + 13}$$

$$= \int \frac{dt}{(t-3)^2 + 2^2}$$

$$= \int \frac{du}{u^2 + 2^2}$$

Complete the square:

$$t^2 - 6t + 13 = (t^2 - 6t) + 13$$

$$= (t^2 - 6t + 9) + 13 - 9$$

$$= (t-3)^2 + 4$$

Let $u = t - 3$

$du = dt$

$$= \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{t-3}{2}\right) + C$$

From Friday

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

(b) $\int \frac{dx}{\sqrt{8x - x^2}}$

Complete the square

$8x - x^2 = -(x^2 - 8x)$

$= -(x^2 - 8x + 16) + 16$

$= -(x-4)^2 + 4^2$

$= \int \frac{dx}{\sqrt{4^2 - (x-4)^2}}$

Let $u = x - 4$

$du = dx$

$= \int \frac{du}{\sqrt{4^2 - u^2}}$

$$= \sin^{-1}\left(\frac{u}{4}\right) + C$$

$$= \sin^{-1}\left(\frac{x-4}{4}\right) + C$$

From Friday

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

(c) $\displaystyle\int \frac{(x+3)\,dx}{\sqrt{x^2+2x+2}}$

Complete the square

$$x^2+2x+2 = (x^2+2x) + 2$$

$$= (x^2+2x+1) + 2 - 1$$

$$= (x+1)^2 + 1$$

$$= \int \frac{x+3}{\sqrt{(x+1)^2+1}}\,dx$$

Let $u = x+1 \Rightarrow x+3 = u+2$

$du = dx$

$$= \int \frac{u+2}{\sqrt{u^2+1}}\,du$$

$$= \int \frac{u}{\sqrt{u^2+1}}\, du + \int \frac{2}{\sqrt{u^2+1}}\, du$$

Integral 1 Integral 2

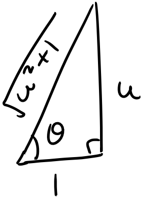
Integral 1: Let $v = u^2+1$, $dv = 2u\,du \Rightarrow \frac{1}{2} dv = u\,du$

$$\int \frac{u}{\sqrt{u^2+1}}\, du = \frac{1}{2}\int v^{-1/2}\, dv$$

$$= \frac{1}{2} \frac{v^{1/2}}{1/2} + C$$

$$= \sqrt{v} + C = \sqrt{u^2+1} + C$$

Integral 2: $\displaystyle\int \frac{2\,du}{\sqrt{u^2+1}}$



$$= \int \frac{2\sec^2\theta\, d\theta}{\sec\theta}$$

$$u = \tan\theta$$

$$\sqrt{u^2+1} = \sec\theta$$

$$= 2\int \sec\theta \, d\theta$$

$$du = \sec^2\theta\, d\theta$$

$$= 2\ln|\sec\theta + \tan\theta| + K$$

$$= 2\ln\left|\sqrt{u^2+1} + u\right| + K$$

So $\int \frac{x+3}{\sqrt{x^2+2x+2}}\,dx$

$u = x+1$

$$= \sqrt{u^2+1} + 2\ln\left|\sqrt{u^2+1} + u\right| + A$$

$$= \sqrt{(x+1)^2+1} + 2\ln\left|\sqrt{(x+1)^2+1} + x+1\right| + A$$

Section 7.5: Rational Functions, Partial Fractions

Simplify

$$\frac{1}{x-3} - \frac{2}{x+4} = \frac{x+4}{(x-3)(x+4)} - \frac{2(x-3)}{(x-3)(x+4)}$$

$$= \frac{x+4-2x+6}{x^2+x-12}$$

$$= \frac{-x+10}{x^2+x-12}$$

Now evaluate the integral

$$\int \frac{10-x}{x^2+x-12}\,dx = \int \left(\frac{1}{x-3} - \frac{2}{x+4} \right) dx$$

$$= \int \frac{1}{x-3}\,dx - 2\int \frac{1}{x+4}\,dx$$

$$= \ln|x-3| - 2\ln|x+4| + C$$

$$* \int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$$

We sort'a cheated! The big question is:

If we started with the simplified *total fraction*

$$\frac{-x+10}{x^2+x-12},$$

how could we figure out that it *decomposes* into the sum of the smaller *partial fractions*

$$\frac{1}{x-3} - \frac{2}{x+4}?$$

Rational Functions

Recall that a rational function is one of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

The function is called a **proper rational function** if

$$\text{degree}(P(x)) \; < \; \text{degree}(Q(x)).$$

Rational Functions

If degree($P(x)$) $\geq$ degree($Q(x)$), then f is an **improper rational function.** In this case, we can write

$$f(x) = p(x) + \frac{r(x)}{Q(x)}$$

where p is a polynomial, and $r(x)/Q(x)$ is proper. We can obtain this using long division.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

Decomposing Proper Rational Functions

Theorem: Every polynomial $Q(x)$ with real coefficients can be factored into a product

$$Q(x) = q_1(x)q_2(x)\cdots q_k(x)$$

where each q_i is either a linear factor (i.e. $q_i(x) = ax + b$) or an irreducible quadratic (i.e. $q_i(x) = ax^2 + bx + c$ where $b^2 - 4ac < 0$).

Knowing that such a factorization exists, and being able to compute it are two different animals! But at least we can know that the cases to be outlined cover all contingencies.

Decomposing Proper Rational Functions

Let $f(x) = P(x)/Q(x)$ be a **proper** rational function, and let $Q(x)$ be factored completely into linear and irreducible quadratic factors

$$f(x) = \frac{P(x)}{q_1(x)q_2(x)\cdots q_k(x)}.$$

We'll consider four cases

(i) each factor of Q is linear and none are repeated,

(ii) each factor of Q is linear and one or more is repeated,

(iii) some factor(s) of Q are quadratic, but no quadratic is repeated,

(iv) Q has at least one repeated quadratic factor.

Case (i) Non-repeated Linear Factors

Suppose $Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$. And no pair of a's and b's (both) match. Then we look for a decomposition of f in the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}.$$

For example

$$\frac{10 - x}{(x-3)(x+4)} = \frac{A}{x-3} + \frac{B}{x+4}.$$

Note that each fraction in the expansion is a proper rational function with denominator a line.

Example: Evaluate the integral

$$\int \frac{4x-2}{x^3-x}\,dx$$

$\dfrac{4x-2}{x^3-x}$ is a proper rational function

partial fraction decomp

$$\frac{4x-2}{x(x^2-1)} = \frac{4x-2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

Clear the fractions

$$x(x-1)(x+1)\,\frac{4x-2}{x(x-1)(x+1)} = \left(\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}\right) x(x-1)(x+1)$$

$$4x - 2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

$$= A(x^2-1) + B(x^2+x) + C(x^2-x)$$

$$= Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx$$

$$0x^2 + 4x - 2 = (A+B+C)x^2 + (B-C)x - A$$

Matching coefficients

$$A + B + C = 0$$

$$B - C = 4$$

$$-A = -2 \implies A = 2$$

From the first equation

$$2+B+C=0 \quad \Rightarrow \quad \begin{aligned} B+C&=-2\\ B-C&=4 \end{aligned}$$

add these

$$2B=2 \Rightarrow B=1$$

subtract

$$2C=-6 \Rightarrow C=-3$$

so

$$\frac{4x-2}{x^3-x}=\frac{2}{x}+\frac{1}{x-1}-\frac{3}{x+1}$$

$$\int \frac{4x-2}{x^3-x}\,dx = \int \left(\frac{2}{x} + \frac{1}{x-1} - \frac{3}{x+1} \right) dx$$

$$= 2\int \frac{1}{x}\,dx + \int \frac{1}{x-1}\,dx - 3\int \frac{1}{x+1}\,dx$$

$$= 2\ln|x| + \ln|x-1| - 3\ln|x+1| + C$$