June 23 Math 2254 sec 001 Summer 2015

Section 7.5: Rational Functions, Partial Fractions

Recall that a rational function is one of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

The function is called a **proper rational function** if

$$\text{degree}(P(x)) \;<\; \text{degree}(Q(x)).$$

Rational Functions

If degree$(P(x)) \geq$ degree$(Q(x))$, then f is an **improper rational function**. In this case, we can write

$$f(x) = p(x) + \frac{r(x)}{Q(x)}$$

where p is a polynomial, and $r(x)/Q(x)$ is proper. We can obtain this using long division.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

Decomposing Proper Rational Functions

Let $f(x) = P(x)/Q(x)$ be a **proper** rational function, and let $Q(x)$ be factored completely into linear and irreducible quadratic factors

$$f(x) = \frac{P(x)}{q_1(x)q_2(x)\cdots q_k(x)}.$$

We'll consider four cases

(i) each factor of Q is linear and none are repeated,

(ii) each factor of Q is linear and one or more is repeated,

(iii) some factor(s) of Q are quadratic, but no quadratic is repeated,

(iv) Q has at least one repeated quadratic factor.

Case (i) Non-repeated Linear Factors

Suppose $Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$. And no pair of a's and b's (both) match. Then we look for a decomposition of f in the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}.$$

For example

$$\frac{10 - x}{(x-3)(x+4)} = \frac{A}{x-3} + \frac{B}{x+4}.$$

Note that each fraction in the expansion is a proper rational function with denominator a line.

Example: Evaluate the integral

$$\int \frac{1}{t^2 - 9}\,dt$$

Partial Fractions:

$$\frac{1}{t^2-9} = \frac{1}{(t-3)(t+3)} = \frac{A}{t-3} + \frac{B}{t+3}$$

$$(t-3)(t+3) \quad \frac{1}{(t-3)(t+3)} = \left(\frac{A}{t-3} + \frac{B}{t+3}\right)(t-3)(t+3)$$

$$1 = A(t+3) + B(t-3)$$

$$1 = At + 3A + Bt - 3B$$

$$\underline{0}t + \underline{\underline{\underline{1}}} = \underline{\underline{(A+B)}}\,t + \underline{\underline{\underline{3A - 3B}}}$$

$$A + B = 0 \implies A = -B$$

$$3A - 3B = 1$$

$$3(-B) - 3B = 1$$

$$-6B = 1 \implies B = \frac{-1}{6}$$

$$A = -B = \frac{1}{6}$$

$$\frac{1}{t^2 - 9} = \frac{1/6}{t-3} + \frac{-1/6}{t+3}$$

$$\int \frac{1}{t^2-9}\,dt = \int \left(\frac{1/6}{t-3} - \frac{1/6}{t+3} \right) dt$$

$$= \frac{1}{6}\int \frac{1}{t-3}\,dt - \frac{1}{6}\int \frac{1}{t+3}\,dt$$

$$= \frac{1}{6}\ln|t-3| - \frac{1}{6}\ln|t+3| + C$$

A simple method for finding coefficients: Nonrepeated Linear Case[1]

From the previous example, we know that

$$\frac{1}{t^2-9} = \frac{A}{t-3} + \frac{B}{t+3} \implies 1 = A(t+3) + B(t-3)$$

Set $t = 3$

$1 = A(3+3) + B(3-3)$

$1 = 6A \Rightarrow A = \frac{1}{6}$

$t = -3$

$1 = A(-3+3) + B(-3-3)$

$1 = -6B \Rightarrow B = \frac{-1}{6}$

[1] This is mentioned on page 501 in Sullivan and Miranda.