June 25 Math 2254 sec 001 Summer 2015

Section 7.5: Rational Functions, Partial Fractions

Let $f(x) = P(x)/Q(x)$ be a **proper** rational function, and let $Q(x)$ be factored completely into linear and irreducible quadratic factors

$$f(x) = \frac{P(x)}{q_1(x)q_2(x)\cdots q_k(x)}.$$

We'll consider four cases

(i) each factor of Q is linear and none are repeated,

(ii) each factor of Q is linear and one or more is repeated,

(iii) some factor(s) of Q are quadratic, but no quadratic is repeated,

(iv) Q has at least one repeated quadratic factor.

Case (ii) A Repeated Linear Factor

Suppose $Q(x)$ has only linear factors, but that one of them is repeated. That is, suppose $(a_i x + b_i)^n$ is a factor of Q. Then for this term, the decomposition of f will contain the n terms

$$\frac{A_{i1}}{a_i x + b_i} + \frac{A_{i2}}{(a_i x + b_i)^2} + \cdots + \frac{A_{in}}{(a_i x + b_i)^n}.$$

For example,

$$\frac{3x^2 + 2x - 1}{(x+1)^2(x-2)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2} + \frac{E}{(x-2)^3}.$$

Example: Evaluate the integral

$$\int \frac{7x^2+7x+4}{x(x+1)^2}\,dx$$

Partial fraction Decomp

Clear fractions

$$x(x+1)^2 \quad \frac{7x^2+7x+4}{x(x+1)^2} = \left(\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}\right) x(x+1)^2$$

$$7x^2+7x+4 = A(x+1)^2 + Bx(x+1) + Cx$$

$$= A(x^2+2x+1) + B(x^2+x) + Cx$$

$$= Ax^2 + 2Ax + A + Bx^2 + Bx + Cx$$

$$\underset{=}{7}x^2 + \underset{=}{7}x + \underset{=}{4} = \underline{\underline{(A+B)}}x^2 + \underline{\underline{(2A+B+C)}}x + \underline{\underline{A}}$$

$$A + B = 7$$
$$2A + B + C = 7$$
$$A = 4 \Rightarrow A = 4$$

$$\Rightarrow B = 7 - A = 7 - 4 = 3$$

$$C = 7 - B - 2A = 7 - 3 - 2(4) = -4$$

$$\boxed{\begin{aligned} A &= 4 \\ B &= 3 \\ C &= -4 \end{aligned}}$$

$$\frac{7x^2 + 7x + 4}{x(x+1)^2} = \frac{4}{x} + \frac{3}{x+1} - \frac{4}{(x+1)^2}$$

$$\int \frac{7x^2+7x+4}{x(x+1)^2}\,dx = \int \left(\frac{4}{x} + \frac{3}{x+1} - \frac{4}{(x+1)^2} \right) dx$$

$$= 4\int \frac{1}{x}\,dx + 3\int \frac{1}{x+1}\,dx - 4\int \frac{1}{(x+1)^2}\,dx$$

Let $u = x+1$
$du = dx$

$$= 4\int \frac{1}{x}\,dx + 3\int \frac{1}{x+1}\,dx - 4\int u^{-2}\,du$$

$$= 4\ln|x| + 3\ln|x+1| - 4\left(\frac{u^{-1}}{-1}\right) + C$$

$$= 4\ln|x| + 3\ln|x+1| + \frac{4}{x+1} + C$$

An Improper Rational Function

Evaluate $\displaystyle \int \frac{2x^3 + x^2 - 6x + 6}{x^2 + x - 2}\, dx$

Do long division to find the polynomial part:

$$
\begin{array}{r|l}
 & 2x - 1 \\
\hline
x^2 + x - 2 & 2x^3 + x^2 - 6x + 6 \\
 & -(2x^3 + 2x^2 - 4x) \\
\hline
 & -x^2 - 2x + 6 \\
 & -(-x^2 - x + 2) \\
\hline
 & -x + 4
\end{array}
$$

$\leftarrow$ polynomial part

$\leftarrow$ remainder

$$\frac{2x^3+x^2-6x+6}{x^2+x-2} = 2x-1 + \underbrace{\frac{-x+4}{x^2+x-2}}$$

Do partial fractions on this part.

$$\frac{-x+4}{x^2+x-2} = \frac{-x+4}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

Clear fractions - mult. by $(x+2)(x-1)$

$$-x+4 = A(x-1) + B(x+2)$$

Use the short cut for nonrepeated linear case

Set $x = 1$ $\quad -1+4 = A(1-x) + B(1+2)$

$$3 = 3B \Rightarrow B = 1$$

Set $x = -2$ $\quad -(-2)+4 = A(-2-1) + B(-2+2)$

$$6 = -3A \Rightarrow A = -2$$

So
$$\frac{2x^3+x^2-6x+6}{x^2+x-2} = 2x-1 + \frac{-2}{x+2} + \frac{1}{x-1}$$

$$\int \frac{2x^3+x^2-6x+6}{x^2+x-2}\,dx = \int \left(2x-1 - \frac{2}{x+2} + \frac{1}{x-1}\right)dx$$

$$= x^2 - x - 2\ln|x+2| + \ln|x-1| + C$$

Case (iii) Nonrepeated Quadratic Factors

Suppose $Q(x)$ has a factor of the form $q(x) = ax^2 + bx + c$ with $(b^2 - 4ac < 0)$ that is not repeated (appears only to the first power). Then for this term, the decomposition of f will contain the term

$$\frac{Ax + B}{ax^2 + bx + c}.$$

For example

$$\frac{3x + 7}{(x+1)(x-2)(x^2+4)(x^2+x+1)} =$$

$$= \frac{A}{x+1} + \frac{B}{x-2} + \frac{Cx + D}{x^2+4} + \frac{Ex + F}{x^2+x+1}.$$

Note that the most **general proper** rational function with a quadratic denominator will have a line in the numerator! It may be that one of A or B is zero, but we don't assume any such thing up front!

Example: Evaluate the Integral

$$\int \frac{3x+4}{(x-1)(x^2+1)}\,dx$$

Partial Fractions

$$\frac{3x+4}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

Clear fractions: mult. by $(x-1)(x^2+1)$

$$3x+4 = A(x^2+1) + (Bx+C)(x-1)$$

$$= Ax^2 + A + Bx^2 + Cx - Bx - C$$

$$\underset{=}{0}x^2 + \underset{=}{3}x + \underset{=}{4} = \underline{\underline{(A+B)}}\,x^2 + \underline{\underline{(-B+C)}}\,x + \underline{\underline{A-C}}$$

$$\left.\begin{array}{l} A+B \qquad = 0 \\ \quad -B + C = 3 \end{array}\right\} \xRightarrow{\text{add}} \left.\begin{array}{l} A + C = 3 \\ A - C = 4 \end{array}\right\} \xRightarrow{\text{add}} 2A = 7$$

$$A \qquad - C = 4$$

$$\boxed{A = \frac{7}{2}}$$

$$\boxed{\begin{array}{l} B = \frac{-7}{2} \\ C = \frac{-1}{2} \end{array}}$$

From 1st eqn: $B = -A = \frac{-7}{2}$

From 3rd eqn: $C = A - 4 = \frac{7}{2} - \frac{8}{2} = \frac{-1}{2}$

So $$\frac{3x+4}{(x-1)(x^2+1)} = \frac{7/2}{x-1} + \frac{-\frac{7}{2}x - \frac{1}{2}}{x^2+1} = \frac{7/2}{x-1} - \frac{\frac{7}{2}x}{x^2+1} - \frac{\frac{1}{2}}{x^2+1}$$

$$\int \frac{3x+4}{(x-1)(x^2+1)}\,dx = \int \frac{7/2}{x-1}\,dx + \int \frac{-\frac{7}{2}x}{x^2+1}\,dx + \int \frac{-\frac{1}{2}}{x^2+1}\,dx$$

Set $u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \frac{7}{2}\int \frac{1}{x-1}\,dx - \frac{7}{2}\cdot\frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$$= \frac{7}{2}\ln|x-1| - \frac{7}{4}\ln|u| - \frac{1}{2}\tan^{-1}x + C$$

$$= \frac{7}{2}\ln|x-1| - \frac{7}{4}\ln|x^2+1| - \frac{1}{2}\tan^{-1}x + C$$

Case (iv) Repeated Quadratic Factor

Suppose $Q(x)$ has a factor of the form $q(x) = (ax^2 + bx + c)^r$ with $(b^2 - 4ac < 0)$ with r an integer bigger than 1. Then for this term, the decomposition of f will contain the r terms

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_r x + B_r}{(ax^2 + bx + c)^r}.$$

For example

$$\frac{1}{(x+1)^2(x^2+4)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+4} + \frac{Ex+F}{(x^2+4)^2} + \frac{Gx+H}{(x^2+4)^3}$$

Example: Evaluate the Integral

$$\int \frac{3x^2 - x + 12}{(x^2+4)^2}\,dx$$

Partial fractions

$$\frac{3x^2 - x + 12}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

Clear fractions: mult. by $(x^2+4)^2$

$$3x^2 - x + 12 = (Ax+B)(x^2+4) + Cx + D$$

$$= Ax^3 + Bx^2 + 4Ax + 4B + Cx + D$$

$$0x^3 + 3x^2 - x + 12 = Ax^3 + Bx^2 + (4A+C)x + 4B + D$$

$$= \quad = \quad = \quad = \quad = \quad = \quad \underline{\underline{}} \quad \underline{\underline{}}$$

$$A = 0 \implies A = 0$$

$$B = 3 \implies B = 3$$

$$4A + C = -1 \implies C = -1 - 4A = -1$$

$$4B + D = 12 \implies D = 12 - 4B = 12 - 12 = 0$$

$$\frac{3x^2 - x + 12}{(x^2+4)^2} = \frac{0x + 3}{x^2+4} + \frac{-x + 0}{(x^2+4)^2} = \frac{3}{x^2+4} - \frac{x}{(x^2+4)^2}$$

$$\int \frac{3x^2-x+12}{(x^2+4)^2}\,dx = \int \frac{3}{x^2+4}\,dx - \int \frac{x}{(x^2+4)^2}\,dx$$

let $u = x^2+4$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= 3\int \frac{dx}{x^2+2^2} - \frac{1}{2}\int u^{-2}\,du$$

$$= 3\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\left(\frac{u^{-1}}{-1}\right) + C$$

$$= \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}\,\frac{1}{x^2+4} + C$$

Recall

$$* \quad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

All Cases: Finding the Right *Form*

Find the form of the partial fraction decomposition (do not solve for any of the coefficients).

(a) $\dfrac{x}{x^2+7x+12} = \dfrac{x}{(x+3)(x+4)} = \dfrac{A}{x+3} + \dfrac{B}{x+4}$

(b) $\dfrac{2x+1}{x^4+x^2} = \dfrac{2x+1}{x^2(x^2+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+1}$

repeated linear factor "x"

(c) $$\frac{x^3}{(x+1)^2(x^2+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$$

(d) $$\frac{x-4}{(x-3)(x^2-9)} = \frac{x-4}{(x-3)(x-3)(x+3)} = \frac{x-4}{(x-3)^2(x+3)}$$

$$= \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x+3}$$