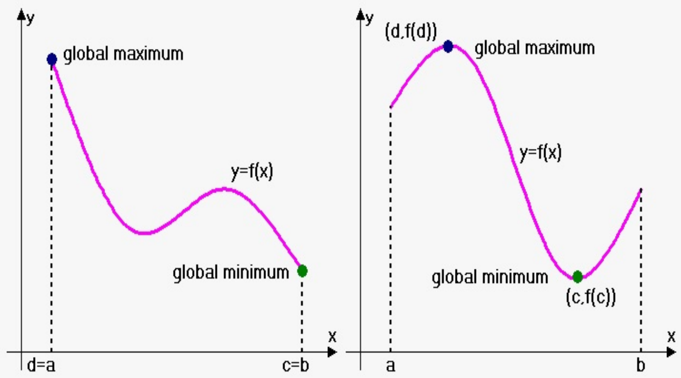
June 26 Math 1190 sec. 51 Summer 2017

Section 4.2: Maximum and Minimum Values; Critical Numbers

Extreme Value Theorem Suppose f is continuous on a closed interval $[a, b]$. Then f attains an absolute maximum value $f(d)$ and f attains an absolute minimum value $f(c)$ for some numbers c and d in $[a, b]$.



Critical Number

Definition: A **critical number** of a function f is a number c in its domain such that either

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

Theorem: If f has a local extremum at c, then c is a critical number of f.

Example

Find all of the critical numbers of the function.

$$g(t) = t^{1/5}(12-t)$$

We did this last time and found that g has two critical numbers. $g'(t)$ was zero when $t = 2$ and $g'(t)$ didn't exist when $t = 0$. Both of these numbers are in the domain of g.

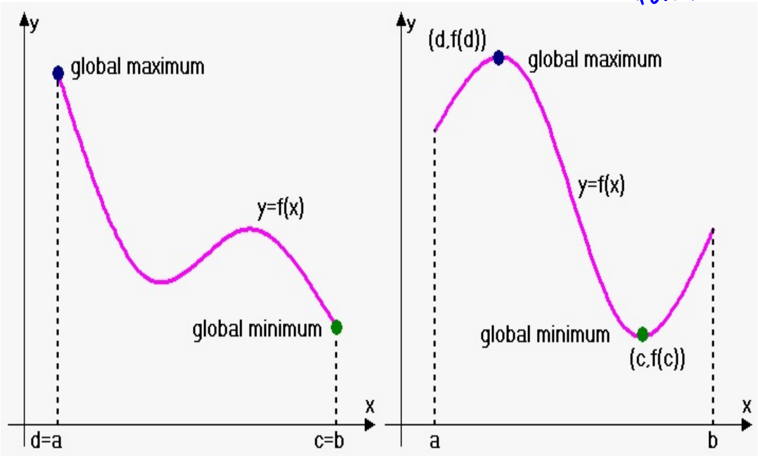
The two critical numbers of g are 0 and 2.

Using the Extreme Value Theorem

When the EVT applies, each absolute extrema occurs either

- at an end point, or
- in between the end points at a critical point.

we just check the list of associated function values



Example

Find the absolute maximum and absolute minimum values of the function on the closed interval.

g is continuous on [-1,1] and [-1,1] is closed.

(a) $g(t) = t^{1/5}(12-t)$, on $[-1, 1]$

We need to compare the function values at the end points and at each critical number between them.

g has two critical numbers, 0 and 2.

Only 0 is in the interval [-1,1].

$$g(t) = t^{1/5}(12-t)$$

We need only compare g at -1, 0, and 1.

$$g(-1) = (-1)^{1/5}\left(12-(-1)\right) = -13$$

$$g(0) = 0^{1/5}(12-0) = 0$$

$$g(1) = 1^{1/5}(12-1) = 11$$

The abs. max value of g is $11 = g(1)$ and the abs. min value of g is $-13 = g(-1)$.

Plot in Walpha

(b) $f(x) = \begin{cases} x\ln x, & x > 0 \\ 0, & x = 0 \end{cases}$ on $[0, e]$

$[0,e]$ is closed.

$x\ln x$ is continuous on $(0,e]$. Is f continuous @ 0?

Is f cont. from the right @ 0?

It is if $\lim_{x\to 0^+} f(x) = f(0) = 0$

$$\lim_{x\to 0^+} x\ln x = \text{"}0\cdot(-\infty)\text{"}$$

we'll use $x\ln x = \dfrac{\ln x}{\frac{1}{x}}$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \text{"}\frac{-\infty}{\infty}\text{"}$$

use l'H rule

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} \frac{1}{x}\cdot\frac{x^2}{-1} = \lim_{x\to 0^+} -x = 0$$

Since $\lim_{x\to 0^+} f(x) = f(0)$, f is cont. from the right

at zero.

Now, we need all critical numbers in $(0, e)$.

$$f'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

$f'(x)$ is undefind nowhere

$$f'(x) = 0 \implies \ln x + 1 = 0$$

$$\ln x = -1$$

$$e^{\ln x} = e^{-1} \implies x = e^{-1} = \frac{1}{e}$$

The number $\frac{1}{e}$ is in the interval — i.e. $0 < \frac{1}{e} < e$.

Now we compare f at the ends and the critical number.

$$f(0) = 0$$

$$f\left(\frac{1}{e}\right) = \frac{1}{e}\ln\frac{1}{e} = \frac{1}{e}(-\ln e) = \frac{-1}{e} \quad \leftarrow \text{abs min}$$

$$f(e) = e\ln e = e\cdot 1 = e \quad \leftarrow \text{abs max}$$

The abs. max value of f is $e = f(e)$, and the abs. min value of f is $\frac{-1}{e} = f\left(\frac{1}{e}\right)$.

Question

Find all of the critical numbers of the function

$$f(x) = 1+27x-x^3$$

$f'(x) = 27 - 3x^2 = 3(9-x^2)$

$= 3(3-x)(3+x)$

(a) 0 and 27

$f'(x)$ - is undefined never

(b) 0 and 3

$f'(x) = 0 \Rightarrow x = 3$ or $x = -3$

(c) -3 and 3

(d) -3, 0, and 3

Question

Find the absolute maximum and absolute minimum values of the function on the closed interval.

$f(0)=1$

$f(3)=55$

$$f(x) = 1+27x-x^3, \quad \text{on} \quad [0,4]$$

$f(4)=45$

(a) Minimum value is 1, maximum value is 55

(b) Minimum value is 1, maximum value is 45

(c) Minimum value is -53, maximum value is 55

(d) Minimum value is -53, maximum value is 45

Section 4.3: The Mean Value Theorem

Rolle's Theorem: Let f be a function that is

i continuous on the closed interval $[a, b]$,
ii differentiable on the open interval (a, b), and
iii such that $f(a) = f(b)$.

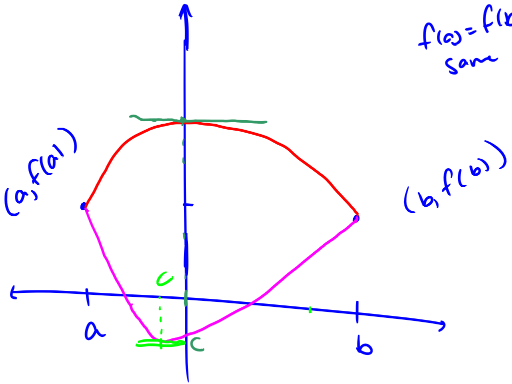
Then there exists a number c in (a, b) such that $f'(c) = 0$.

The Mean Value Theorem (MVT) is arguably the most significant theorem in calculus. This is even accounting for a theorem we'll discuss later called the *Fundamental Theorem of Calculus*.

Rolle's Theorem

One possibility: $f(x) = k$ a constant

In that case, $f'(c) = 0$ for all c in (a,b).



$f(a) = f(b)$
same y-value

If f goes up from $(a, f(a))$ it must come down. If it goes down it must come up. Where it turns, it has a horizontal tangent.

Example

Show that the function $f(\theta) = \cos\theta + \sin\theta$ has at least one point c in $\left[0, \frac{\pi}{2}\right]$ such that $f'(c) = 0$.

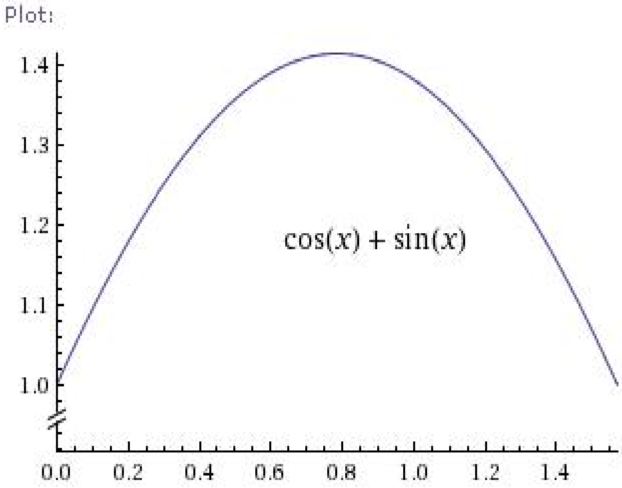
f is continuous and differentiable everywhere.

so f is continuous on $[0, \pi/2]$, f is differentiable on $(0, \frac{\pi}{2})$.

Also $f(0) = \cos 0 + \sin 0 = 1 + 0 = 1$

and $f(\frac{\pi}{2}) = \cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 0 + 1 = 1$

Rolle's theorem guarantees that for some c in $(0, \pi/2)$, $f'(c) = 0$.



Figure

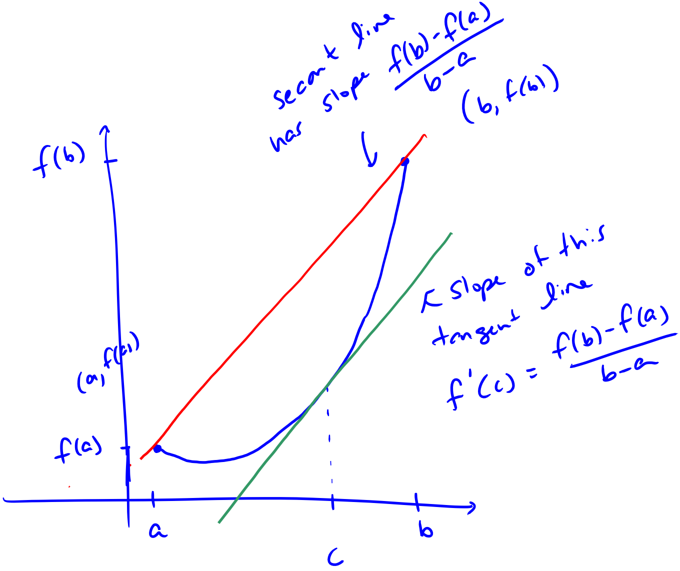
The Mean Value Theorem

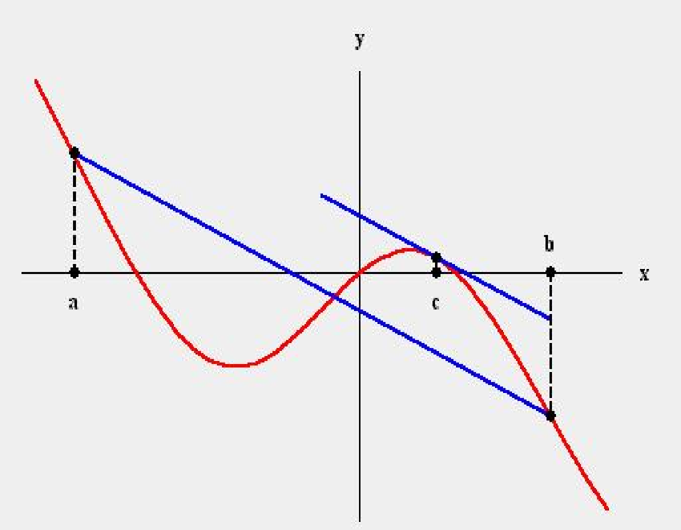
Theorem: Suppose f is a function that satisfies

i f is continuous on the closed interval $[a, b]$, and
ii f is differentiable on the open interval (a, b).

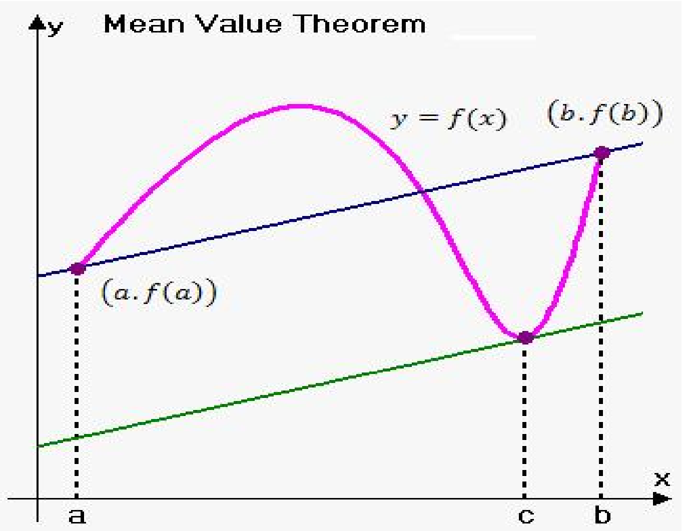
Then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad \text{equivalently} \quad f(b) - f(a) = f'(c)(b - a).$$


secant line
has slope $\frac{f(b)-f(a)}{b-a}$
$(b, f(b))$
$f(b)$
$(a, f(a))$
$f(a)$
a
c
b
slope of this
tangent line
$f'(c) = \frac{f(b)-f(a)}{b-a}$



Figure



Figure

Figure: Celebration of the MVT in Beijing.

Example

Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all values of c that satisfy the conclusion of the MVT.

$$f(x) = x^3 - 2x, \quad [0, 2]$$

As a polynomial, f is differentiable on $(-\infty, \infty)$. So

f is continuous on $[0, 2]$ and

f is differentiable on $(0, 2)$.

Here, $a = 0$ and $b = 2$. we're looking for all c's

such that $f'(c) = \dfrac{f(2) - f(0)}{2 - 0}$

$f(x) = x^3 - 2x \Rightarrow f(2) = 2^3 - 2\cdot 2 = 4$ and $f(0) = 0^3 - 2\cdot 0 = 0$

and $f'(x) = 3x^2 - 2$. Our equation is

$$f'(c) = 3c^2 - 2 = \frac{4-0}{2-0} = 2$$

$$\Rightarrow \quad 3c^2 = 4$$

$$\Rightarrow \quad c^2 = \frac{4}{3} \Rightarrow c = \sqrt{\frac{4}{3}} \text{ or } c = -\sqrt{\frac{4}{3}}$$

The solution $\sqrt{\frac{4}{3}}$ is the only one in the interval $(0, 2)$.

Important Consequence of the MVT

Theorem: If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Corollary: If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b). In other words,

$$f(x) = g(x) + C \quad \text{where } C \text{ is some constant.}$$

Examples

Find all possible functions $f(x)$ that satisfy the condition

(a) $f'(x) = \cos x$ on $(-\infty, \infty)$

We need one example function $g(x) = \sin x$

So all such functions must be

$f(x) = \sin x + C$ when C is any constant.

(b) $f'(x) = 2x$ on $(-\infty, \infty)$

An example is $g(x) = x^2$. All such functions

are

$$f(x) = x^2 + C$$ for constant C.

Question

Find all possible functions $h(t)$ that satisfy the condition

$$h'(t) = \sec^2 t \quad \text{on} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

(a) $h(t) = \sec^2 t + C$

(b) $h(t) = \tan t + 1$

(c) $h(t) = \tan t + C$

Another Consequence of the MVT

Another significant consequence of the MVT is that it provides a test for the increasing and decreasing behavior of a differentiable function.

Theorem: Let f be differentiable on an open interval (a, b). If

- $f'(x) > 0$ on (a, b), the f is increasing on (a, b), and
- $f'(x) < 0$ on (a, b), the f is decreasing on (a, b).

Example

Determine the intervals over which f is increasing and the intervals over which it is decreasing where

$$f(x) = 2x^3 - 6x^2 - 18x + 1$$

The domain of f is $(-\infty, \infty)$. We want to know where $f'(x) > 0$ and where $f'(x) < 0$. We'll look for where $f'(x)$ can change signs. These are where $f'(x) = 0$ or $f'(x)$ is undefined. So we need the critical #s.

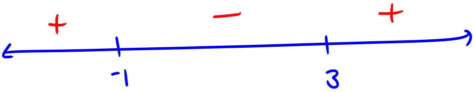
$$f'(x) = 6x^2 - 12x - 18 = 6(x^2 - 2x - 3) = 6(x-3)(x+1)$$

$f'(x)$ is never undefined.

$f'(x) = 0 \implies 6(x-3)(x+1) = 0 \implies x = 3$ or $x = -1$.

These critical numbers divide the domain into 3 parts: $(-\infty, -1)$, $(-1, 3)$, and $(3, \infty)$.

We can do a sign analysis of f'.



$f'(x) = 6(x-3)(x+1)$

$(-\infty,-1)$	test pt.	-2	$f'(-2)=6(-2-3)(-2+1)$	$+$
$(-1,3)$	test pt	0	$f'(0)=6(0-3)(0+1)$	$-$
$(3,\infty)$	test pt	4	$f'(4)=6(4-3)(4+1)$	$+$

From our analysis, f is increasing on

$(-\infty,-1)\cup(3,\infty)$.

f is decreasing on $(-1,3)$.

Question

Suppose that we compute the derivative of some function g and find

$$g'(x) = (2+x)e^{x/2}.$$

$g'(x)=0 \Rightarrow x=-2$

$g'(x)$ UND Never

Determine the intervals over which g is increasing and over which it is decreasing.

$-$ $\quad$ $+$

-2

(a) g is increasing on $(-1/2, \infty)$ and decreasing on $(-\infty, -1/2)$.

(b) g is increasing on $(-2, \infty)$ and decreasing on $(-\infty, -2)$.

(c) g is increasing on $(2, \infty)$ and decreasing on $(-\infty, 2)$.

(d) g is increasing on $(-\infty, -2)$ and decreasing on $(-2, \infty)$.

Section 4.4: Local Extrema and Concavity

We have already seen that the first derivative f' can tell us about the behaviour of the function f—in particular, it gives information about where it is increasing or decreasing, and where it may take a local extreme value.

In this section, we'll expand on that as well as introduce information about a function that can be deduced from the nature of its second derivative.

Theorem: First derivative test for local extrema

Let f be continuous and suppose that c is a critical number of f.

- If f' changes from negative to positive at c, then f has a local minimum at c.

- If f' changes from positive to negative at c, then f has a local maximum at c.

- If f' does not change signs at c, then f does not have a local extremum at c.

Note: we read from left to right as usual when looking for a sign change.

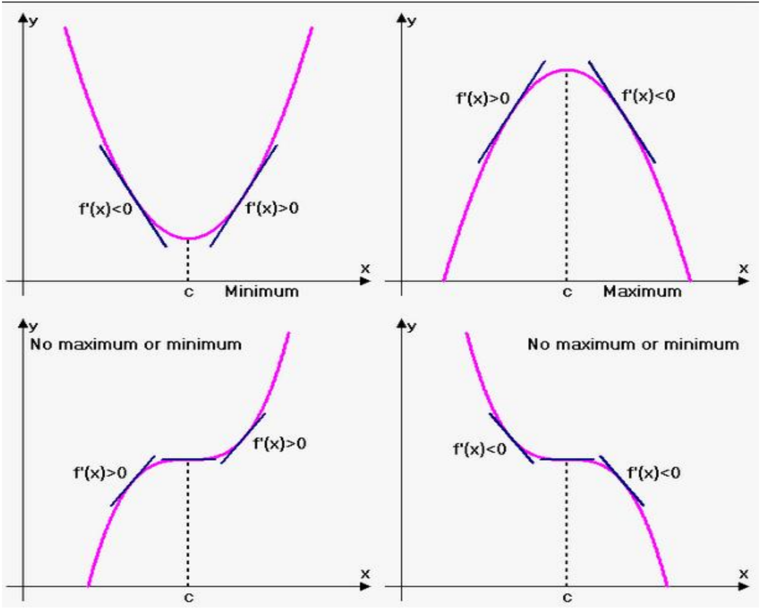


Figure: First derivative test

Example

Find all the critical points of the function and classify each one as a local maximum, a local minimum, or neither.

$$f(x) = x^{1/3}(16 - x) = 16x^{1/3} - x^{4/3}$$

The domain is $(-\infty, \infty)$. Find all critical #s.

$$f'(x) = 16\left(\frac{1}{3}x^{-2/3}\right) - \frac{4}{3}x^{1/3}$$

$$= \frac{16}{3x^{2/3}} - \frac{4x^{1/3}}{3}$$

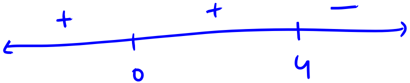
$$= \frac{16}{3x^{2/3}} - \frac{4x^{1/3}}{3} \cdot \frac{x^{2/3}}{x^{2/3}} = \frac{16 - 4x}{3x^{2/3}}$$

$$f'(x) = \frac{4(4-x)}{3x^{2/3}}$$

$$f'(x) = 0 \Rightarrow 4(4-x) = 0 \Rightarrow x = 4$$

$$f'(x) \text{ is UND} \Rightarrow 3x^{2/3} = 0 \Rightarrow x = 0$$

We can run a sign analysis on $f'(x)$.



$$f'(x) = \frac{4(4-x)}{3\sqrt[3]{x^2}}$$

Test pt. -1 $\quad f'(-1) = \dfrac{4(4-(-1))}{3\sqrt[3]{(-1)^2}}$ $\quad \dfrac{+}{+}$ $\quad +$

Test pt 1 $\quad f'(1) = \dfrac{4(4-1)}{3\sqrt[3]{1^2}}$ $\quad \dfrac{+}{+}$ $\quad +$

Test pt. 5 $\quad f'(5) = \dfrac{4(4-5)}{3\sqrt[3]{5^2}}$ $\quad \dfrac{-}{+}$ $\quad -$

f has two critical numbers, 0 and 4.

f has neither a local max nor min at 0.

f has a local maximum at 4.

Question

Find all of the critical numbers of $f(t) = t^4 + 4t^3$.

(a) 0, 3, and -3

$f'(t) = 4t^2(t+3)$

(b) 3 and -3

(c) 0 and -3

(d) Can't be determined without more information.

Question

Consider the function $f(t) = t^4 + 4t^3$. Which of the following is true about this function?

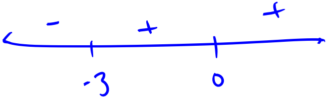
(a) f has a local minimum at $t = 0$ and a local maximum at $t = -3$.

(b) f has a local minimum at $t = -3$ and a local maximum at $t = 0$.

$f'(t) = 4t^2(t+3)$

(c) f has a local minimum at $t = -3$.

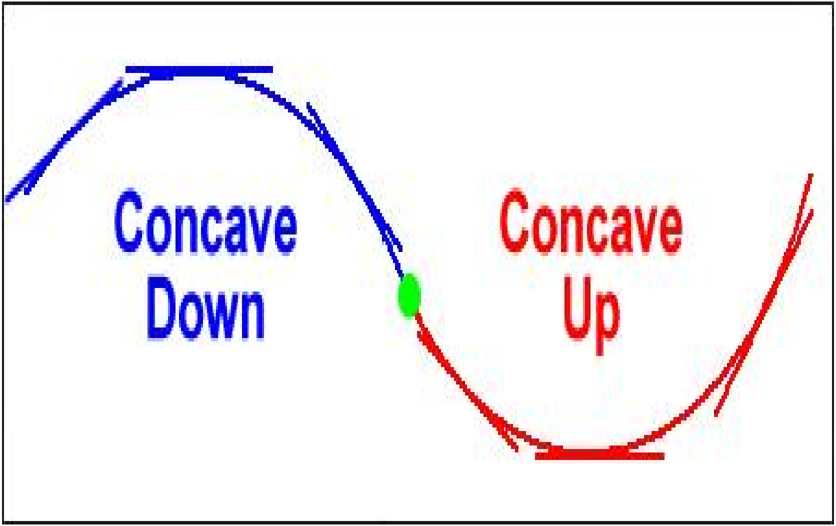
(d) f has a local minimum at $t = 0$.



Concavity and The Second Derivative

Concavity: refers to the *bending* nature of a graph. In particular, a curve is concave down if it's cupped side is down, and it is concave up if it's cupped upward.

Concavity



Figure

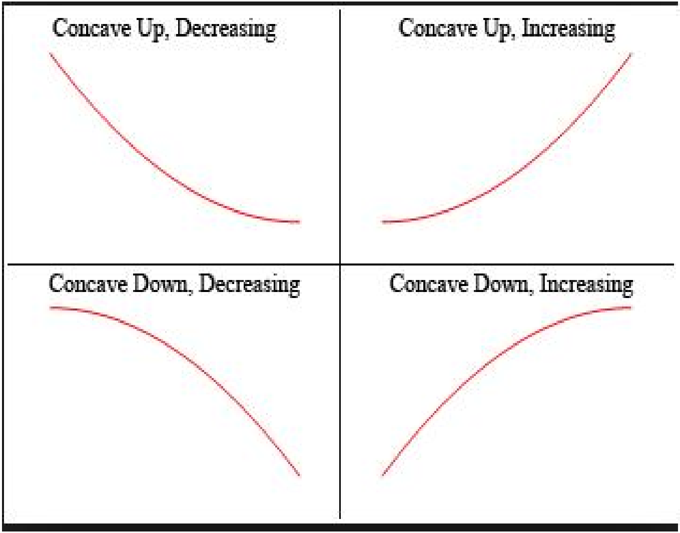


Figure: A graph can have either increasing or decreasing behavior and be either concave up or down.

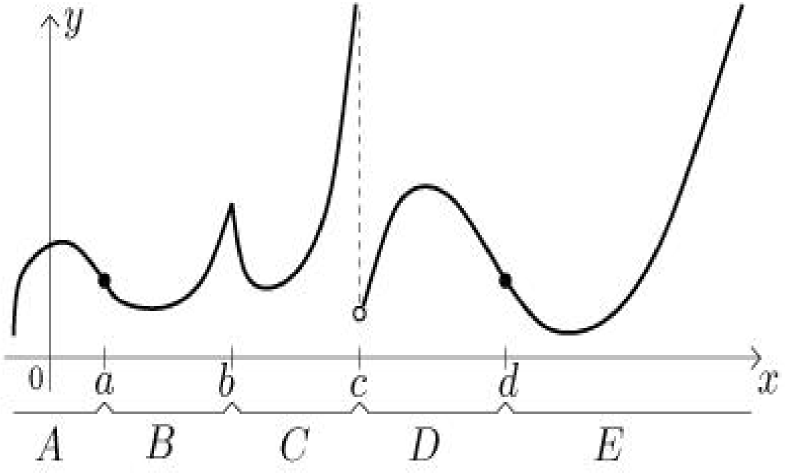


Figure: We can consider concavity at a point, but it's best thought of as a property over an interval. Many function's graphs have concavity that changes over the domain.

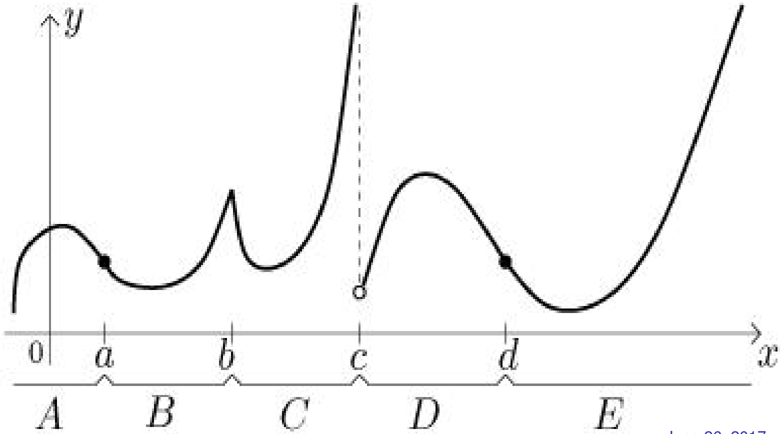
Definition of Concavity

If the graph of a function f lies above all of its tangent lines over an interval I, then f is concave up on I. If the graph of f lies below each of its tangent lines on an interval I, f is concave down on I.

Theorem: (Second Derivative Test for Concavity)
Suppose f is twice differentiable on an interval I.

- If $f''(x) > 0$ on I, then the graph of f is concave up on I.
- If $f''(x) < 0$ on I, then the graph of f is concave down on I.

Definition: A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous at P and the concavity of f changes at P (from down to up or from up to down). A point where $f''(x) = 0$ would be a candidate for being an inflection point.



Concavity and Extrema:

Theorem: (Second Derivative Test for Local Extrema)
Suppose $f'(c) = 0$ and that f'' is continuous near c. Then

- if $f''(c) > 0$, f takes a local minimum at c,
- if $f''(c) < 0$, then f takes a local maximum at c.

If $f''(c) = 0$, then the test fails. f may or may not have a local extrema. You can go back to the first derivative test to find out.

Example

Analyze the function $f(x) = xe^{3x}$. In particular, indicate

- the intervals on which f is increasing and decreasing,
- the intervals on which f is concave up and concave down,
- identify critical points and classify any local extrema, and
- identify any points of inflection.

We'll need f' and f''.

$$f'(x) = 1\cdot e^{3x} + x\left(e^{3x}\cdot 3\right) = e^{3x}(1+3x)$$

$$f''(x) = 3e^{3x}(1+3x) + e^{3x}(3) = e^{3x}(3+9x+3) = e^{3x}(6+9x)$$

Find critical points:

$$f'(x) = 0 \Rightarrow e^{3x}(1+3x) = 0$$

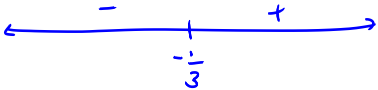
$$\Rightarrow e^{3x} = 0 \quad \text{or} \quad 1+3x = 0$$

no soln. $\qquad x = \frac{-1}{3}$

$f'(x)$ is UND $\Rightarrow$ no solutions

f has one critical number $\frac{-1}{3}$.

Sign analysis on f'



test pt. -1, $f'(-1) = e^{-3}(1-3)$ $-$

test pt. 0, $f'(0) = e^{0}(1+0)$ $+$

f is increasing on $(\frac{-1}{3}, \infty)$, decreasing on $(-\infty, \frac{-1}{3})$ and has a local minimum at $x = \frac{-1}{3}$.

We'll finish later.