June 26 Math 2254 sec 001 Summer 2015

Section Section 7.8: Improper Integrals

Consider the function $f(t) = \frac{1}{t^2}$. Note that this function is never negative.

What is wrong with the following statement?

$$\int_{-2}^{1} \frac{1}{t^2}\,dt = -\frac{1}{t}\Big|_{-2}^{1} = -\frac{1}{1} - \left(-\frac{1}{-2}\right) = -\frac{3}{2}$$

The FTC doesn't apply because f is not continuous on $[-2, 1]$.

Improper Integrals

The integral

$$\int_a^b f(x)\,dx$$

is **improper** if a and/or b is infinite (i.e. $a = -\infty$, $b = \infty$ or both), or if f has an infinite discontinuity at a, b, or somewhere between them (i.e. the graph of f has a vertical asymptote).

The integral **may or may not** have a well defined value. The Fundamental Theorem of Calculus **does not** apply!

Type 1: $\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^b f(x)\,dx$

Definition: Suppose $\int_a^t f(x)\,dx$ exists for every number $t \geq a$. Then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided the limit exists (as a finite number).

Similarly, if $\int_t^b f(x)\,dx$ exists for every number $t \leq b$. Then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided the limit exists (as a finite number).

Definition Continued...

In either case, if the limit exists, then the integral is said to be **convergent**. Otherwise, it is **divergent**.

If both limits are infinite, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

for any real c provided both integrals on the right are convergent.

Example: Horn of Gabriel

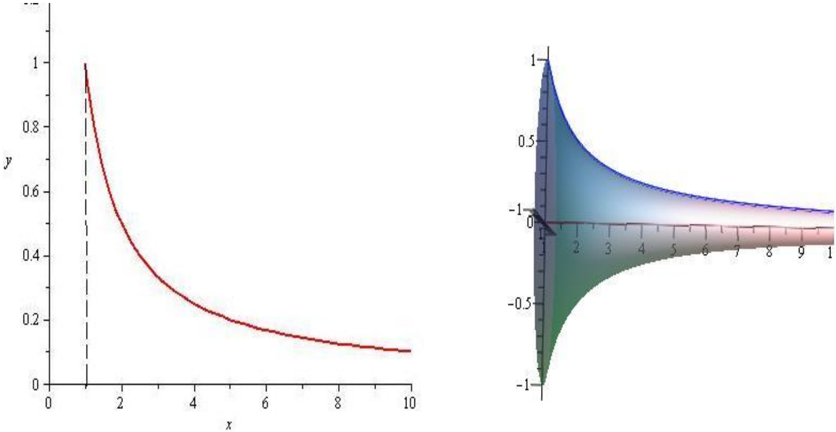
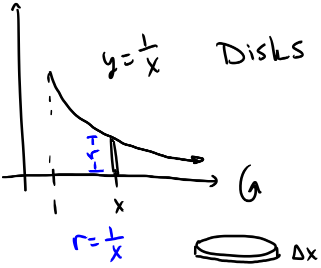


Figure: Consider the region under the curve $f(x) = \frac{1}{x}$ for $1 \leq x < \infty$. Let this be rotated about the x-axis.

Example: Horn of Gabriel

Show that the horn of Gabriel is finite.



$$V = \int_1^\infty \frac{\pi}{x^2}\,dx$$

$$= \lim_{t\to\infty} \int_1^t \frac{\pi}{x^2}\,dx$$

$$= \lim_{t\to\infty} \pi \int_1^t x^{-2}\,dx$$

$$V_{Disk} = \pi r^2 \Delta x$$

$$= \pi\left(\frac{1}{x}\right)^2 \Delta x$$

$$= \frac{\pi}{x^2}\,\Delta x$$

$$= \lim_{t\to\infty} \pi \frac{x^{-1}}{-1}\Big|_1^t$$

$$= \lim_{t\to\infty} \frac{-\pi}{x}\Big|_1^t = \lim_{t\to\infty}\left[\frac{-\pi}{t} - \frac{-\pi}{1}\right]$$

$$= 0 + \pi$$

$$= \pi$$

The volume of the horn is π.

The integral $\int_1^\infty \frac{\pi}{x^2}\,dx$ is convergent.

Evaluate the Improper Integral if Possible

(a) $\int_1^\infty \frac{dx}{x}$ $= \lim_{t\to\infty} \int_1^t \frac{dx}{x}$

$= \lim_{t\to\infty} \left(\ln|x| \Big|_1^t \right.$

$= \lim_{t\to\infty} \left(\ln|t| - \ln|1| \right)$

$= \infty - 0 = \infty$

This integregl is divergent.