Section 8.1: The Law of Cosines

Theorem: For the triangle labeled using the previous convention, all three of the following equations hold

\[ a^2 = b^2 + c^2 - 2bc \cos A \]
\[ b^2 = a^2 + c^2 - 2ac \cos B \]
\[ c^2 = a^2 + b^2 - 2ab \cos C \]

This can be used if two sides and an included angle (SAS) are known, or if the three sides (SSS) are known.
The Law of Cosines & The Pythagorean Theorem

Apply the law of cosines to a right triangle for which $C = 90^\circ$; see what it produces.

Only known is $C = 90^\circ$

\[ c^2 = a^2 + b^2 - 2ab \cos C \]
\[ c^2 = a^2 + b^2 - 2ab \cos 90^\circ \]
\[ c^2 = a^2 + b^2 - 0 \]
\[ c^2 = a^2 + b^2 \]
Example (SSS)

Solve the triangle given \( a = 5, \ b = 2, \ c = 6 \)

**Law of Cosines**

\[
C^2 = a^2 + b^2 - 2ab \cos C \\
6^2 = 5^2 + 2^2 - 2(5)(2) \cos C \\
36 = 25 + 4 - 20 \cos C \implies \\
20 \cos C = 29 - 36 = -7 \\
\cos C = \frac{-7}{20} \\
C = \cos^{-1}\left(\frac{-7}{20}\right) \approx 110.5^\circ
\]
Given \( c = 6 \) and \( C = 110.5^\circ \), we can use the law of sines.

Law of sines
\[
\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow \frac{\sin A}{5} = \frac{\sin 110.5^\circ}{6}
\]

\( \sin A = \frac{5 \sin 110.5^\circ}{6} \Rightarrow A = \sin^{-1} \left( \frac{5 \sin (110.5^\circ)}{6} \right) \approx 51.3^\circ \)
\[ B = 180^\circ - A - C \]

\[ = 180^\circ - 51.3^\circ - 110.5^\circ = 18.2^\circ \]

\[ A = 51.3^\circ \quad a = 5 \]

\[ B = 18.2^\circ \quad b = 2 \]

\[ C = 110.5^\circ \quad c = 6 \]
Area of a Triangle

**Theorem:** The area of a triangle is one half the product of any two sides times the sine of the included angle. That is

\[
\text{Area} = \frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B = \frac{1}{2} bc \sin A.
\]

There is an alternative theorem that can be used if no angles are known.

**Theorem: (Heron's Formula)** For the triangle with sides of lengths \(a\), \(b\), and \(c\). Define the semi-perimeter

\[
s = \frac{a + b + c}{2}.
\]

The area of the triangle is

\[
\text{Area} = \sqrt{s(s - a)(s - b)(s - c)}.
\]
Example

A set designer needs to estimate the amount of paint required to paint a triangular piece of wooden backdrop. Determine the area of the wood piece shown to the nearest tenth of a square meter.

Let

\[ a = 4 \, \text{m} \]
\[ b = 7 \, \text{m} \]
\[ c = 9 \, \text{m} \]

\[ S = \frac{a+b+c}{2} = \frac{4+7+9}{2} = 10 \, \text{m} \]
Use Heron's formula

\[ s = 10n \]
\[ S - a = 10n - 4n = 6n \]
\[ S - b = 10n - 7n = 3n \]
\[ S - c = 10n - 9n = 1n \]

\[
\text{area} = \sqrt{s(s-a)(s-b)(s-c)}
\]
\[
= \sqrt{10n(6n)(3n)(1n)} = \sqrt{180n^4}
\]
\[
= 13.4 \text{ m}^2
\]
Section 8.2: Polar Coordinates & Polar Equations

Given a point in the plane, we can completely characterize it (relative to an origin) with an ordered pair 

$$(x, y).$$
Pole, Polar Axis, Polar Coordinates

Figure: Note the correct notation is (distance first, angle second).
Connecting Polar to Rectangular Coordinates

\[ \cos \theta = \frac{x}{r} \]

\[ x = r \cos \theta \]

\[ \sin \theta = \frac{y}{r} \]

\[ y = r \sin \theta \]

\[ r^2 = x^2 + y^2 \]

\[ \tan \theta = \frac{y}{x} \text{ for } x \neq 0 \]
Plot \((\sqrt{2}, \frac{3\pi}{4})\) and \((2, -\frac{2\pi}{3})\)

\[ x = \sqrt{2} \cos \left( \frac{3\pi}{4} \right) = \frac{-\sqrt{2}}{\sqrt{2}} = -1 \]

\[ y = \sqrt{2} \sin \left( \frac{3\pi}{4} \right) = \frac{\sqrt{2}}{\sqrt{2}} = 1 \]

\[ (-1, 1) = (x, y) \]

\[ x = 2 \cos \left( \frac{-2\pi}{3} \right) = \frac{-2}{2} = -1 \]

\[ y = 2 \sin \left( \frac{-2\pi}{3} \right) = \frac{-2\sqrt{3}}{2} = -\sqrt{3} \]

\[ (-1, -\sqrt{3}) = (x, y) \]
Negative Values of $r$

If $r > 0$, then the point $(-r, \theta) = (r, \theta + \pi)$.

Plot the points $(-\sqrt{2}, \frac{3\pi}{4})$, and $(-2, -\frac{2\pi}{3})$. 

We can call this $\left(\sqrt{2}, -\frac{\pi}{4}\right)$.
Polar Coordinates are not Unique

Find two other coordinate pairs for the same point. For one, take \( r < 0 \). For the second, choose \( \theta \) in the interval \([2\pi, 4\pi)\).

(a) \( (3, \frac{\pi}{3}) \)

For \( r < 0 \):

\( (-3, \frac{4\pi}{3}) \)

For \( 2\pi \leq \theta < 4\pi \):

\( (3, \frac{7\pi}{3}) \)
Polar Coordinates are not Unique

Find two other coordinate pairs for the same point. For one, take $r < 0$. For the second, choose $\theta$ in the interval $[2\pi, 4\pi)$.

(b) \( \left( \sqrt{6}, \frac{7\pi}{6} \right) \)

\[
\begin{align*}
r < 0 & : \quad \left( -\sqrt{6}, \frac{13\pi}{6} \right) \\
2\pi \leq \theta < 4\pi & : \quad \left( \sqrt{6}, \frac{19\pi}{6} \right)
\end{align*}
\]
Converting between Coordinate Systems

The coordinates \((x, y)\) are called **rectangular** or **Cartesian**\(^1\) coordinates.

\[
x = r \cos \theta, \quad y = r \sin \theta
\]

\[
x^2 + y^2 = r^2, \quad \tan \theta = \frac{y}{x} \quad \text{for } x \neq 0
\]

If \(x = 0\), then \(\theta = \frac{\pi}{2}\) or \(\theta = -\frac{\pi}{2}\)—of course any co-terminal \(\theta\) may be used.

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\(^1\)Named in honor of René Descartes, the *father* of Analytic Geometry.
Examples

Express the polar coordinates in Cartesian coordinates.

(a) \( \left( 2, \frac{\pi}{6} \right) = \left( \sqrt{3}, 1 \right) \)

\[
\begin{align*}
X &= 2 \cos \frac{\pi}{6} = 2 \left( \frac{\sqrt{3}}{2} \right) = \sqrt{3} \\
Y &= 2 \sin \frac{\pi}{6} = 2 \left( \frac{1}{2} \right) = 1
\end{align*}
\]

(b) \( \left( -3, \frac{7\pi}{3} \right) = \left( -\frac{3}{2}, -\frac{3\sqrt{3}}{2} \right) \)

\[
\begin{align*}
X &= -3 \cos \left( \frac{7\pi}{3} \right) = -3 \left( \frac{1}{2} \right) = -\frac{3}{2} \\
Y &= -3 \sin \left( \frac{7\pi}{3} \right) = -3 \left( \frac{\sqrt{3}}{2} \right) = -\frac{3\sqrt{3}}{2}
\end{align*}
\]
(c) \((4, \pi) = (-4, 0)\)

\[
\begin{align*}
  x &= y \cos \pi = 4(-1) = -4 \\
  y &= y \sin \pi = 4(0) = 0
\end{align*}
\]

(d) \((14, 0.35) = (13.15, 4.80)\)

\[
\begin{align*}
  x &= 14 \cos (0.35) \approx 13.15 \\
  y &= 14 \sin (0.35) \approx 4.80
\end{align*}
\]
Examples

Express the Cartesian point in polar coordinates. (Choose \( r > 0 \) and \( 0 \leq \theta < 2\pi \).)

(a) \((1, -1)\) = \((\sqrt{2}, \frac{7\pi}{4})\)

\[
\begin{align*}
\rho &= \sqrt{1^2 + (-1)^2} = \sqrt{2} \\
\tan \theta &= \frac{-1}{1} = -1 \\
\tan^{-1} (-1) &= -\frac{\pi}{4}
\end{align*}
\]

To get \( 0 \leq \theta < 2\pi \), use \( \theta = \frac{7\pi}{4} \).
(b) \((-1, 1) = \left(\sqrt{2}, \frac{3\pi}{4}\right)\)

\[
c = \sqrt{(-1)^2 + 1^2} = \sqrt{2}
\]

\[
\tan \theta = \frac{1}{-1} = -1
\]

\[
\tan^{-1}(-1) = -\frac{\pi}{4}
\]

Need the point in Quadrant II and

\[0 \leq \theta < 2\pi\]

Take \(\theta = \frac{3\pi}{4}\)
(c) \((-3\sqrt{3}, -3) = (6, \frac{7\pi}{6})\)

\[
c = \sqrt{(-3\sqrt{3})^2 + (-3)^2} = \sqrt{27 + 9} = \sqrt{36} = 6
\]

\[
\tan \theta = -\frac{3}{-3\sqrt{3}} = \frac{1}{\sqrt{3}}
\]

\[
\tan^{-1} \left( \frac{1}{\sqrt{3}} \right) = \frac{\pi}{6}
\]

we need quad III with \(0 < \theta < 2\pi\)

\[
\tan \theta = \frac{7\pi}{6}
\]
(d) \((-2, 7) = (\sqrt{53}, \tan^{-1}(-\frac{7}{2}) + \pi)\)

\[\approx (7.28, 1.85)\]

\[r = \sqrt{(-2)^2 + 7^2} = \sqrt{4 + 49} = \sqrt{53}\]

\[\tan \theta = -\frac{7}{2} \quad \tan^{-1}(-\frac{7}{2})\]

we need \textbf{Quad II} - add \pi

\[\theta = \tan^{-1}(-\frac{7}{2}) + \pi\]
Polar Graphs

Functions in polar coordinates generally appear in the form\(^2\)

\[ r = f(\theta). \]

**Special Cases:** Determine the nature of the graph of the equation in polar coordinates

(a) \( r = 2, \)

all points are 2 units away from the origin.

It's a circle of radius 2 centered at \((0,0)\)

(b) \( \theta = \frac{\pi}{4} \)

A line connecting \(r = 0\) point to \((0,0)\) makes a \(45^\circ\) with the x-axis.

It's the \(45^\circ\) line \((y = x)\)

\[ \tan \theta = \tan \frac{\pi}{4} = \frac{y}{x} \Rightarrow y = x \]

\(^2\)Note the analogy to function of the form \(y = f(x)\)
Polar Graphs

Analyze the function expressed in polar coordinates. Plot its graph by converting the equation to Cartesian coordinates.

\[ r = 4 \sin \theta \]

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]

and

\[ x^2 + y^2 = r^2 \]

\[ r = y \sin \theta \implies r^2 = yr \sin \theta \]

\[ x^2 + y^2 = 4y \implies x^2 + y^2 - 4y = 0 \]

Complete the square

\[ x^2 + (y^2 - 4y + 4) = 4 \]
\[ x^2 + (y - 2)^2 = 2^2 \]

Circle of radius 2 centered at \((0, 2)\)