June 28 Math 1190 sec. 51 Summer 2017

Section 4.4: Local Extrema and Concavity

We recall the theorem: If f takes a local extremum at c, then c is a critical number of f.

A result following from the Mean Value Theorem told us that if $f'(x) > 0$ on an interval, f is increasing on that interval. Similarly, if $f'(x) < 0$ on an interval, f is decreasing on that interval.

This gives rise to the first derivative test.

Theorem: First derivative test for local extrema

Let f be continuous and suppose that c is a critical number of f.

- If f' changes from negative to positive at c, then f has a local minimum at c.
- If f' changes from positive to negative at c, then f has a local maximum at c.
- If f' does not change signs at c, then f does not have a local extremum at c.

Note: we read from left to right as usual when looking for a sign change.

Concavity and The Second Derivative

Then, we defined concavity: If the graph of a function f lies above all of its tangent lines over an interval I, then f is concave up on I. If the graph of f lies below each of its tangent lines on an interval I, f is concave down on I.

Theorem: (Second Derivative Test for Concavity)
Suppose f is twice differentiable on an interval I.

- If $f''(x) > 0$ on I, then the graph of f is concave up on I.
- If $f''(x) < 0$ on I, then the graph of f is concave down on I.

Definition: A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous at P and the concavity of f changes at P.

Concavity and Extrema:

Theorem: (Second Derivative Test for Local Extrema)
Suppose $f'(c) = 0$ and that f'' is continuous near c. Then

- if $f''(c) > 0$, f takes a local minimum at c,
- if $f''(c) < 0$, then f takes a local maximum at c.

If $f''(c) = 0$, then the test fails. f may or may not have a local extrema. You can go back to the first derivative test to find out.

Example

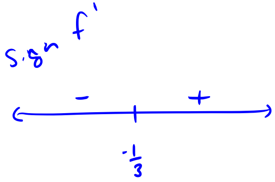
Analyze the function $f(x) = xe^{3x}$. In particular, indicate

- the intervals on which f is increasing and decreasing,
- the intervals on which f is concave up and concave down,
- identify critical points and classify any local extrema, and
- identify any points of inflection.

We were in the middle of this example. Now we will finish it out.

We found the first two derivatives to be

$$\begin{aligned} f'(x) &= e^{3x}(1+3x), \quad \text{and} \\ f''(x) &= e^{3x}(6+9x). \end{aligned}$$



We analyzed $f'(x)$ and determined that

- f is increasing on $(-1/3, \infty)$,
- f is decreasing on $(-\infty, -1/3)$, and
- f has a local minimum taken at $-1/3$.

$$f''(x) = 3e^{3x}(2+3x)$$

Where could f'' change signs?

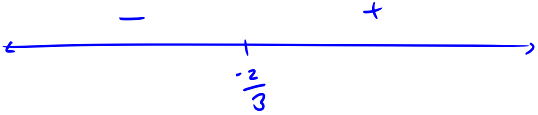
When is $f''(x)$ is undefined? Never f'' is always defined.

When is $f''(x) = 0$? $0 = 3e^{3x}(2+3x)$

$\Rightarrow$ $3e^{3x} = 0$ which has no solution

or $2+3x = 0$ $\Rightarrow$ $x = \frac{-2}{3}$

Sign analysis of $f''(x) = 3e^{3x}(2+3x)$



Test pt -1, $f''(-1) = 3e^{-3}(2-3)$ $(+)(-)$ $-$

0, $f''(0) = 3e^{0}(2+0)$ $(+)(+)$ $+$

f is concave down on $(-\infty, \frac{-2}{3})$ and concave up on $(\frac{-2}{3}, \infty)$. f has an inflection point at $\left(\frac{-2}{3}, f\left(\frac{-2}{3}\right)\right)$.

Note what the 2nd derivative test says about the critical number $\frac{-1}{3}$.

$$f''\left(\frac{-1}{3}\right) = 3e^{3\left(\frac{-1}{3}\right)}\left(2+3\left(\frac{-1}{3}\right)\right)$$

$$= 3e^{-1}(2-1) = 3e^{-1} = \frac{3}{e} > 0$$

This says that f has a local min at $x = \frac{-1}{3}$.

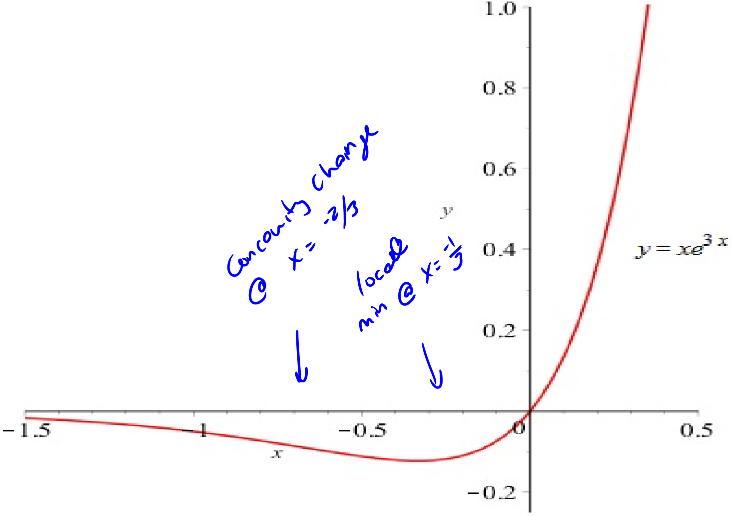


Figure: Plot of $y = xe^{3x}$.

Question

(1) **True or False** If $f''(2) = 0$ it must be that f has an inflection point $(2, f(2))$.

Concavity may or may not change.

Conside the example $f(x) = (x-2)^4$

Question

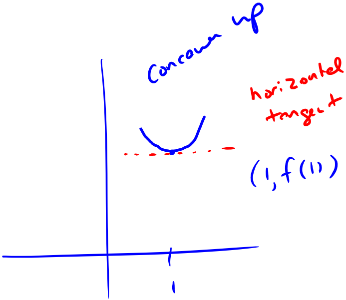
(2) Suppose that we know a function f satisfies the two conditions $f'(1) = 0$ and $f''(1) = 4$. Which of the following can we conclude with certainty?

(a) f has a local minimum at $(1, f(1))$.

(b) f has an inflection point at $(1, f(1))$.

(c) f has a local maximum at $(1, f(1))$.

(d) None of the above are necessarily true.



Section 4.8: Antiderivatives; Differential Equations

Definition: A function F is called an antiderivative of f on an interval I if

$$F'(x) = f(x) \quad \text{for all } x \text{ in } I.$$

For example, $F(x) = x^2$ is an antiderivative of $f(x) = 2x$ on $(-\infty, \infty)$. Similarly, $G(x) = \tan x + 7$ is an antiderivative of $g(x) = \sec^2 x$ on $(-\pi/2, \pi/2)$.

Theorem: If F is any antiderivative of f on an interval I, then the *most general* antiderivative of f on I is

$$F(x) + C \quad \text{where } C \text{ is an arbitrary constant.}$$

Find the most general antiderivative of f.

(i) $f(x) = \cos x \quad I = (-\infty, \infty)$

Since $\frac{d}{dx} \sin x = \cos x$

$F(x) = \sin x + C$ for any constant C.

(ii) $f(x) = \dfrac{1}{x} \quad I = (0, \infty)$

$F(x) = \ln x + C$

Question: Find the most general antiderivative of f.

(iii) $f(x) = \sin x \quad I = (-\infty, \infty)$

Recall $\frac{d}{dx} \cos x = -\sin x$

so $\frac{d}{dx}(-\cos x) = -(-\sin x)$

$= \sin x$

(a) $F(x) = \cos x$

(b) $F(x) = \cos x + C$

(c) $F(x) = -\cos x + C$

Question: Find the most general antiderivative of f.

(iv) $f(x) = \sec x \tan x \quad I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(a) $F(x) = \sec x$

(b) $F(x) = \sec x + C$

(c) $F(x) = \tan x + C$

Find the most general antiderivative of

$$f(x) = x^n, \quad \text{where } n = 1, 2, 3, \ldots$$

Let's suppose $F(x) = Ax^k$ where the numbers A and k are to be determined.

Then $F'(x) = A(kx^{k-1}) = kAx^{k-1}$

So it must be that

$$kAx^{k-1} = x^n$$

Matching the coefficient and the power, we get

$$kA = 1$$

$$k - 1 = n$$

$k-1=n \Rightarrow k=n+1$, $kA=1 \Rightarrow (n+1)A=1$

so $A = \dfrac{1}{n+1}$

so $F(x) = \dfrac{1}{n+1} x^{n+1}$ or more generally

$F(x) = \dfrac{1}{n+1} x^{n+1} + C$ for constant C.

This is usually written as

$$\frac{x^{n+1}}{n+1}$$

Some general results[1]:

(See the table on page 330 in Sullivan & Miranda for a more comprehensive list.)

Function	Particular Antider.	Function	Particular Antider.
$cf(x)$	$cF(x)$	$\cos x$	$\sin x$
$f(x)+g(x)$	$F(x)+G(x)$	$\sin x$	$-\cos x$
$x^n,\ n\neq -1$	$\frac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\csc x\cot x$	$-\csc x$
$\frac{1}{x^2+1}$	$\tan^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$

[1] We'll use the term particular antiderivative to refer to any antiderivative that has no arbitrary constant in it.

Example

Find the most general antiderivative of $h(x) = x\sqrt{x}$ on $(0, \infty)$.

Note $h(x) = x\sqrt{x} = x\,x^{1/2} = x^{3/2}$

$$H(x) = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C = \frac{x^{5/2}}{5/2} + C$$

$$\boxed{H(x) = \frac{2}{5}x^{5/2} + C}$$

Example

Determine the function $H(x)$ that satisfies the following conditions

$$H'(x) = x\sqrt{x}, \quad \text{for all } x > 0, \text{ and } H(1) = 0.$$

From the last example, H has the form

$H(x) = \frac{2}{5} x^{5/2} + C$, $H(1) = 0 \Rightarrow$ H = 0 when x = 1

Imposing $H(1) = 0$, $H(1) = \frac{2}{5}(1)^{5/2} + C = 0$

$\frac{2}{5} + C = 0 \Rightarrow C = \frac{-2}{5}$. The solution is

$$\boxed{H(x) = \frac{2}{5} x^{5/2} - \frac{2}{5}.}$$

Example

A particle moves along the x-axis so that its acceleration at time t is given by

$$a(t) = 12t - 2 \quad \text{m/sec}^2.$$

At time $t = 0$, the velocity v and position s of the particle are known to be

$$v(0) = 3 \quad \text{m/sec, and} \qquad s(0) = 4 \text{ m.}$$

Find the position $s(t)$ of the particle for all $t > 0$.

$a(t) = v'(t) \Rightarrow$ v is an antiderivative of a.

$$v(t) = 12 \cdot \frac{t^{1+1}}{1+1} - 2 \cdot t + C = 12\frac{t^2}{2} - 2t + C$$

$$v(t) = 6t^2 - 2t + C$$

$$v(0)=3 \Rightarrow \quad 3=6\cdot 0^2-2\cdot 0+C \Rightarrow \quad C=3$$

So $\quad v(t)=6t^2-2t+3$

Since $\quad v(t)=s'(t)$, s is an antiderivative of v.

$$s(t)=6\,\frac{t^{2+1}}{2+1}-2\,\frac{t^{1+1}}{1+1}+3t+C$$

$$=6\,\frac{t^3}{3}-2\,\frac{t^2}{2}+3t+C$$

$$=2t^3-t^2+3t+C$$

Since $s(0) = 4$

$$4 = 2\cdot 0^3 - 0^2 + 3\cdot 0 + C$$

$$\Rightarrow C = 4$$

The position function is

$$s(t) = 2t^3 - t^2 + 3t + 4.$$

Example

A **differential equation** is an equation that involves the derivative(s) of an unknown function. **Solving** such an equation would mean finding such an unknown function.

Solve the differential equation subject to the given *initial* conditions.

$$\frac{d^2y}{dx^2} = \cos x + 2, \quad y(0) = 0, \quad y'(0) = -1$$

Find $\frac{dy}{dx}$: $\quad \frac{dy}{dx} = \sin x + 2x + C$

Find y : $\quad y = -\cos x + 2\frac{x^{1+1}}{1+1} + Cx + D$

So $y'(x) = \sin x + 2x + C$

and $y(x) = -\cos x + x^2 + Cx + D$

with $y(0) = 0$ and $y'(0) = -1$

$$0 = y(0) = -\cos 0 + 0^2 + C\cdot 0 + D$$

$$0 = -1 + D \implies D = 1$$

$$-1 = y'(0) = \sin 0 + 2\cdot 0 + C \implies -1 = C$$

The solution is

$$\boxed{y = -\cos x + x^2 - x + 1.}$$

Section 5.1: Area (under the graph of a nonnegative function)

We will investigate the area enclosed by the graph of a function f. We'll make the following assumptions (for now):

- f is continuous on the interval $[a,b]$, and
- f is nonnegative, i.e $f(x) \geq 0$, on $[a,b]$.

Our Goal: Find the area of such a region.

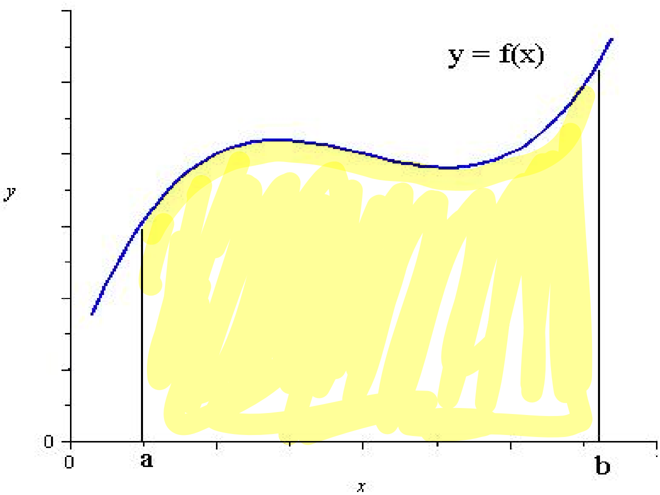


Figure: Region under a positive curve $y = f(x)$ on an interval $[a, b]$.

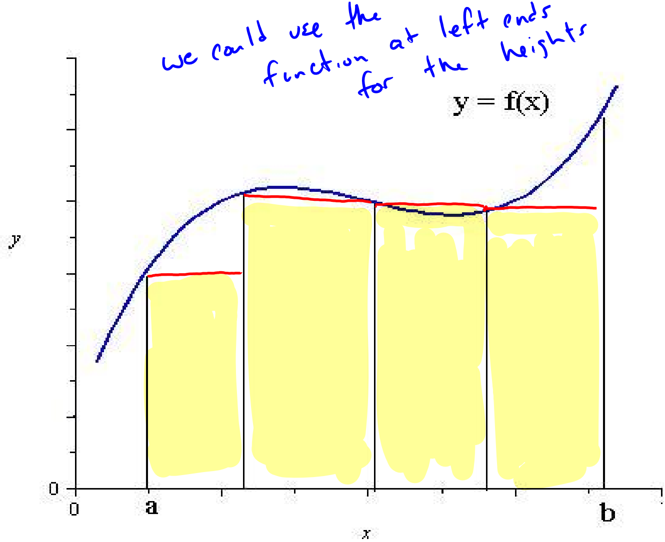


Figure: We could approximate the area by filling the space with rectangles.

we could use right ends of intervals

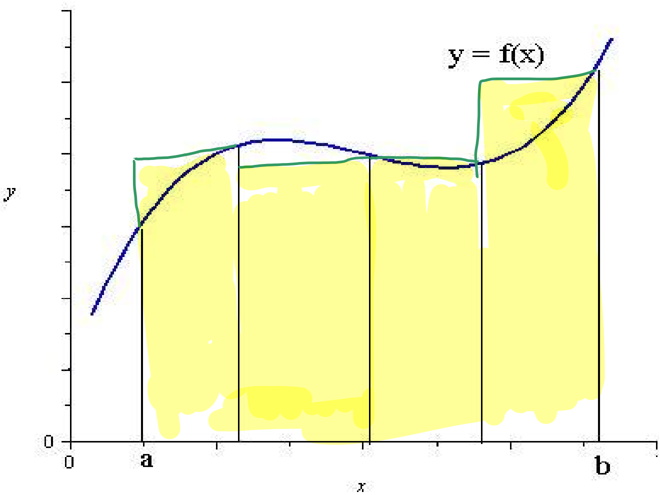


Figure: We could approximate the area by filling the space with rectangles.

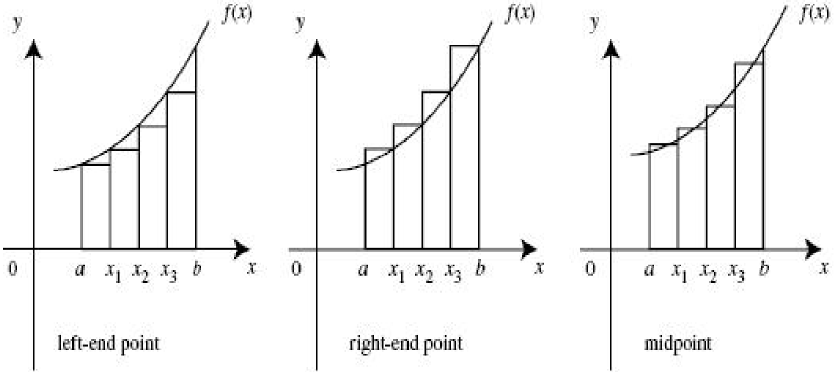


Figure: Some choices as to how to define the heights.

Approximating Area Using Rectangles

We can experiment with

- Which points to use for the heights (left, right, middle, other....)
- How many rectangles we use

to try to get a good approximation.

Definition: We will define the true area to be value we obtain taking the limit as the number of rectangles goes to $+\infty$.

Some terminology

- A **Partition** P of an interval $[a, b]$ is a collection of points $\{x_0, x_1, ..., x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

- A **Subinterval** is one of the intervals $x_{i-1} \leq x \leq x_i$ determined by a partition.
- The width of a subinterval is denoted $\Delta x_i = x_i - x_{i-1}$. If they are all the same size (equal spacing), then

$$\Delta x = \frac{b-a}{n}, \quad \text{and this is called the } \textbf{norm} \text{ of the partition.}$$

- A set of **sample points** is a set $\{c_1, c_2, \ldots, c_n\}$ such that $x_{i-1} \leq c_i \leq x_i$.

Taking the number of rectangles to ∞ is the same as taking the width $\Delta x \to 0$.

Example:

Write an equally spaced partition of the interval $[0, 2]$ with the specified number of subintervals, and determine the norm Δx.

(a) For $n = 4$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2}$$

$$x_0 = a, \quad x_n = b$$

$$x_1 = x_0 + \Delta x$$

$$x_1 = 0 + \frac{1}{2} = \frac{1}{2}$$

$$x_2 = x_1 + \Delta x = \frac{1}{2} + \frac{1}{2} = 1$$

$$x_3 = x_2 + \Delta x = 1 + \frac{1}{2} = \frac{3}{2}$$

$$x_4 = x_3 + \Delta x = \frac{3}{2} + \frac{1}{2} = \frac{4}{2} = 2$$

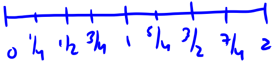
$$\left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}$$

Example:

Write an equally spaced partition of the interval $[0, 2]$ with the specified number of subintervals, and determine the norm Δx.

(b) For $n = 8$

$\Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{8} = \dfrac{1}{4}$



$x_0 = 0$

$x_1 = x_0 + \Delta x = 0 + \frac{1}{4} = \frac{1}{4}$

$\left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2\right\}$

Question

Write an equally spaced partition of the interval $[0, 2]$ with 6 subintervals, and determine the norm Δx.

(a) $\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\}$ $\quad \Delta x = \frac{1}{3}$

(b) $\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\}$ $\quad \Delta x = \frac{1}{6}$

(c) $\{0, \frac{1}{6}, \frac{1}{3}, 1, \frac{5}{6}, \frac{7}{6}, 2\}$ $\quad \Delta x = \frac{1}{3}$

(c) Find an equally spaced partition of $[0, 2]$ having N subintervals. What is the norm Δx?

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{N} = \frac{2}{N}$$

$$x_0 = 0$$

$$x_1 = x_0 + \Delta x = \frac{2}{N}$$

$$x_2 = x_1 + \Delta x = (x_0 + \Delta x) + \Delta x = x_0 + 2\Delta x = 2\left(\frac{2}{N}\right)$$

$$x_3 = x_2 + \Delta x = (x_0 + 2\Delta x) + \Delta x = x_0 + 3\Delta x = 3\left(\frac{2}{N}\right)$$

$$x_4 = x_0 + 4\Delta x = 4\left(\frac{2}{N}\right)$$

In general, $x_i = x_0 + i\Delta x$ for $i = 1, 2, 3, \dots, N$

Note $x_n = x_N = x_0 + N\Delta x = 0 + N\left(\frac{2}{N}\right) = 2$

So the partition is $\{x_0, x_1, \dots, x_N\}$

where $x_i = x_0 + i\Delta x$, $i = 0, \dots, N$

and $\Delta x = \frac{2}{N}$

Approximating area with a Partition and sample points

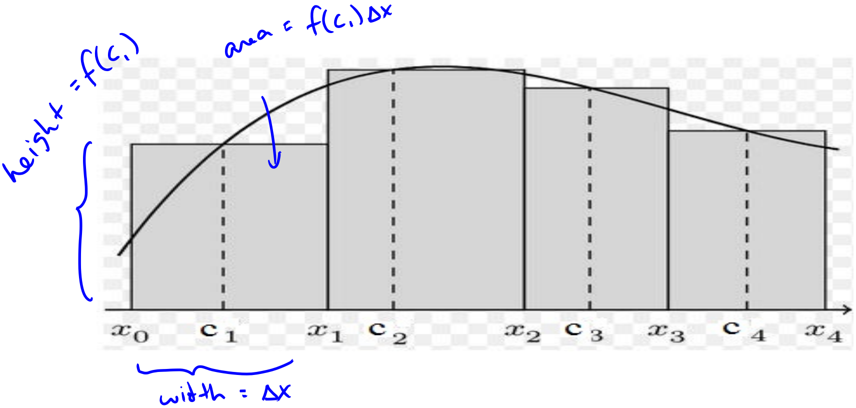


Figure: Area = $f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + f(c_4)\Delta x$. This can be written as

$$\sum_{i=1}^{n} f(c_i)\Delta x.$$

Sum Notation

$\sum$ is the capital letter *sigma*, basically a capital Greek "S".

If $a_1, a_2, \ldots, a_n$ are a collection of real numbers, then

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

This is read as

the sum from i equals 1 *to n of* a_i *(a sub i).*

For example

$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$$

$$\sum_{i=1}^{3} 2i^2 = 2\cdot 1^2 + 2\cdot 2^2 + 2\cdot 3^2 = 2 + 8 + 18 = 28$$

In general, an equally spaced partition of $[a, b]$ with n subintervals means

- $\Delta x = \frac{b-a}{n}$
- $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, i.e. $x_i = a + i\Delta x$
- Taking heights to be

$$\text{left ends} \quad c_i = x_{i-1} \quad \text{area} \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$\text{right ends} \quad c_i = x_i \quad \text{area} \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

- The true area exists (for f continuous) and is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

Lower and Upper Sums

The standard way to set up these sums is to take c_i such that

$$f(c_i) \quad \text{is the abs. minimum value of } f \text{ on } [x_{i-1}, x_i]$$

Then set A_L

$$A_L = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

This is called a **Lower Riemann sum**.

Lower and Upper Sums

Then, we take C_i such that

$$f(C_i) \quad \text{is the abs. maximum value of } f \text{ on } [x_{i-1}, x_i]$$

Then set A_U

$$A_U = \lim_{n\to\infty} \sum_{i=1}^{n} f(C_i)\Delta x.$$

This is called a **Upper Riemann sum**.

Lower and Upper Sums

If f is continuous on $[a, b]$, then it will necessarily be that

$$A_L = A_U.$$

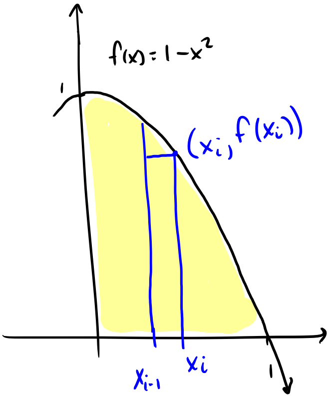
This value is the true area.

In practice, these are tough to compute unless f is only increasing or only decreasing. So instead, we tend to use left and right sums.

Example: Find the area under the curve $f(x) = 1 - x^2$, $0 \le x \le 1$.

Use right end points $c_i = x_i$ and assume the following identity

$$\sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + n}{6} \quad \text{(sum of first } n \text{ squares)}$$



Impose a partion with norm $\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$

$x_0 = 0$

$x_i = x_0 + i\Delta x = 0 + i\left(\frac{1}{n}\right) = i\left(\frac{1}{n}\right)$

For one representative rectangle, the area

$$A_i = \underbrace{f(x_i)}_{\text{height}} \underbrace{\Delta x}_{\text{width}} = (1 - x_i^2)\frac{1}{n}$$

$$= \left(1 - \left(i\left(\frac{1}{n}\right)\right)^2\right)\frac{1}{n}$$

$$= \left(1 - i^2\left(\frac{1}{n}\right)^2\right)\frac{1}{n} = \frac{1}{n} - i^2\left(\frac{1}{n}\right)^3$$

With n rectangles

$$A \approx \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(\frac{1}{n} - i^2\left(\frac{1}{n}\right)^3\right)$$

We'll simplify, then take $n \to \infty$.

$$\sum_{i=1}^{n} \left(\frac{1}{n} - i^2 \left(\frac{1}{n}\right)^3 \right)$$

$$= \sum_{i=1}^{n} \frac{1}{n} - \sum_{i=1}^{n} i^2 \left(\frac{1}{n}\right)^3$$

$$= \frac{1}{n} \sum_{i=1}^{n} 1 - \frac{1}{n^3} \sum_{i=1}^{n} i^2$$

$$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ times}} = n$$

using the given formula for $\sum_{i=1}^{n} i^2$

So

$$A \approx \frac{1}{n} \cdot n - \frac{1}{n^3}\left(\frac{2n^3 + 3n^2 + n}{6}\right)$$

$$= 1 - \frac{2n^3 + 3n^2 + n}{6n^3}$$

Now we find A by taking the limit

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n \to \infty} \left(1 - \frac{2n^3 + 3n^2 + n}{6n^3}\right)$$

$$= \lim_{n\to\infty} \left(1 - \left(\frac{2n^3}{6n^3} + \frac{3n^2}{6n^3} + \frac{n}{6n^3}\right)\right)$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{3} - \frac{1}{2n} - \frac{1}{6n^2}\right)$$

$$= 1 - \frac{1}{3} - 0 - 0 = \frac{2}{3}$$

Recovering Distance from Velocity

The speedometer readings for a motorcycle are recorded at 12 second intervals. Use the information in the table to estimate the total distance traveled. Get estimates using

(a) left end points (beginning time of intervals), and
(b) right end points (ending time for each interval).

t in sec	0	12	24	36	48	60
v in ft/sec	20	28	25	22	24	27

t in sec	0	12	24	36	48	60
v in ft/sec	20	28	25	22	24	27

- left

- right

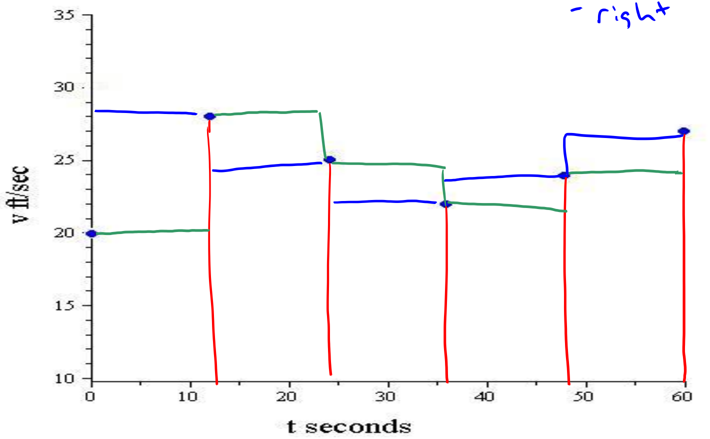


Figure: Graphical representation of motorcycle's velocity.

t in sec	0	12	24	36	48	60
v in ft/sec	20	28	25	22	24	27

$$D \approx 20\,\frac{\text{ft}}{\text{sec}} \cdot 12\text{ sec} + 28\,\frac{\text{ft}}{\text{sec}} \cdot 12\text{ sec}$$

$$+ 25\,\frac{\text{ft}}{\text{sec}} \cdot 12\text{ sec} + 22\,\frac{\text{ft}}{\text{sec}} \cdot 12\text{ sec} + 24\,\frac{\text{ft}}{\text{sec}} \cdot 12\text{ sec}$$

$$= (20+28+25+22+24) \cdot 12 \text{ ft}$$

$$= 1428 \text{ ft}$$

t in sec	0	12	24	36	48	60
v in ft/sec	20	28	25	22	24	27

$$D \approx 28\,\frac{\text{ft}}{\text{sec}}\cdot 12\,\text{sec} + 25\,\frac{\text{ft}}{\text{sec}}\cdot 12\,\text{sec} + 22\,\frac{\text{ft}}{\text{sec}}\cdot 12\,\text{sec}$$

$$+ 24\,\frac{\text{ft}}{\text{sec}}\cdot 12\,\text{sec} + 27\,\frac{\text{ft}}{\text{sec}}\cdot 12\,\text{sec}$$

$$= (28+25+22+24+27)\cdot 12\ \text{ft}$$

$$= 1512\ \text{ft}$$

Our Motorcycle's True Velocity is Probably "Smooth"

True distance is probably between 1428 ft and 1512 ft

*Note: The exact distance is the area of this shaded region!

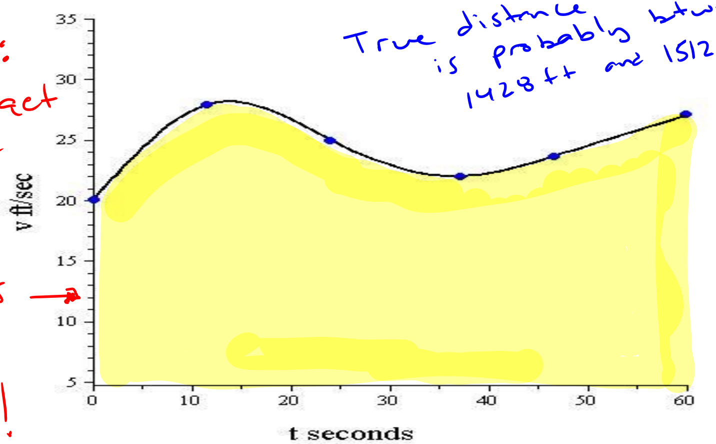


Figure: The true graph of the velocity probably looks more like this. But we only know for certain what it is at the recorded times.