June 29 Math 2254 sec 001 Summer 2015

Section Section 7.8: Improper Integrals

The integral
$$\int_a^b f(x)\,dx$$
is **improper** if a and/or b is infinite (i.e. $a=-\infty$, $b=\infty$ or both), or if f has an infinite discontinuity at a, b, or somewhere between them (i.e. the graph of f has a vertical asymptote).

The integral **may or may not** have a well defined value. The Fundamental Theorem of Calculus **does not** apply!

Type 1: $\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^b f(x)\,dx$

Definition: Suppose $\int_a^t f(x)\,dx$ exists for every number $t \geq a$. Then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided the limit exists (as a finite number).

Similarly, if $\int_t^b f(x)\,dx$ exists for every number $t \leq b$. Then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided the limit exists (as a finite number).

Definition Continued...

In either case, if the limit exists, then the integral is said to be **convergent**. Otherwise, it is **divergent**.

If both limits are infinite, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

for any real c provided both integrals on the right are convergent.

Evaluate the Improper Integral if Possible

(b) $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{x^2+1} + \int_{0}^{\infty} \frac{dx}{x^2+1}$$

$$\int_{-\infty}^{0} \frac{dx}{x^2+1} = \lim_{t\to-\infty} \int_{t}^{0} \frac{dx}{x^2+1}$$

$$= \lim_{t\to-\infty} \tan^{-1} x \Big|_{t}^{0}$$

$$= \lim_{t\to-\infty} \left(\tan^{-1} 0 - \tan^{-1} t\right) = 0 - \left(\frac{-\pi}{2}\right) = \frac{\pi}{2},$$

So the integral is convergent with value $\frac{\pi}{2}$.

$$\int_0^\infty \frac{dx}{x^2+1} = \lim_{t\to\infty} \int_0^t \frac{dx}{x^2+1}$$

$$= \lim_{t\to\infty} \tan^{-1} x \Big|_0^t$$

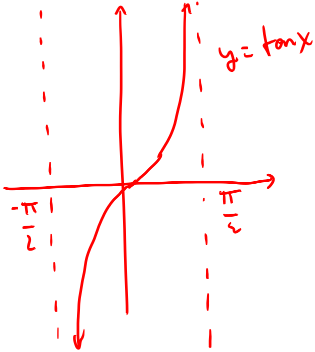
$$= \lim_{t\to\infty} \left[\tan^{-1} t - \tan^{-1} 0\right] = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

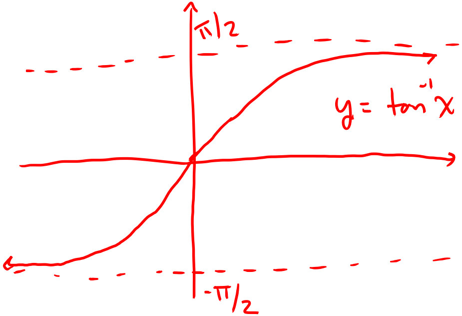
This integral converges with value $\pi/2$.

Hence $\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$ converges and

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$\lim_{y\to\infty} \tan^{-1} y = \frac{\pi}{2} \quad \text{and} \quad \lim_{y\to-\infty} \tan^{-1} y = \frac{-\pi}{2}$$





Determine the values of p for which $\int_1^\infty \frac{dx}{x^p}$ converges.

Recall from Friday (6/26/15) that the integral $\int_1^\infty \frac{dx}{x}$ is divergent—this is the $p = 1$ case.

let $p \neq 1$

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx$$

$$= \lim_{t\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^t$$

$$= \lim_{t\to\infty} \left(\frac{t^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1} \right)$$

2 cases : ① $-p+1>0$ and ② $-p+1<0$

① $-p+1>0 \Rightarrow p<1$ (the power on t is positive)

$$\lim_{t\to\infty} \frac{t^{-p+1}}{-p+1} = \infty$$ the integral diverges

② $-p+1<0 \Rightarrow p>1$ (the power on t is negative)

$$\lim_{t\to\infty} \frac{t^{-p+1}}{-p+1} = \lim_{t\to\infty} \frac{1}{(-p+1)\,t^{p-1}} = 0$$

In this case, the integral converges and

$$\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty}\left(\frac{t^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}\right)$$

$$= 0 - \frac{1}{-p+1} = \frac{1}{p-1}$$

for $p>1$

General Result [1]

$$\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1} \quad \text{if } p > 1\text{. It is divergent if } \; p \leq 1.$$

[1] We'll use this again, so keep it in mind!

Type 2: $\int_a^b f(x)\,dx$ with f discontinuous

Definition: If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

provided the limit exists (as a finite number).

Similarly, if f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

provided the limit exists (as a finite number).

Definition Continued...

Again, if the limit exists in either case, the integral is said to be **convergent**. Otherwise it is **divergent**.

If f is continuous on $[a, b]$ except at some point c in (a, b)—-i.e. it is continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

provided each of the integrals on the right side are convergent.

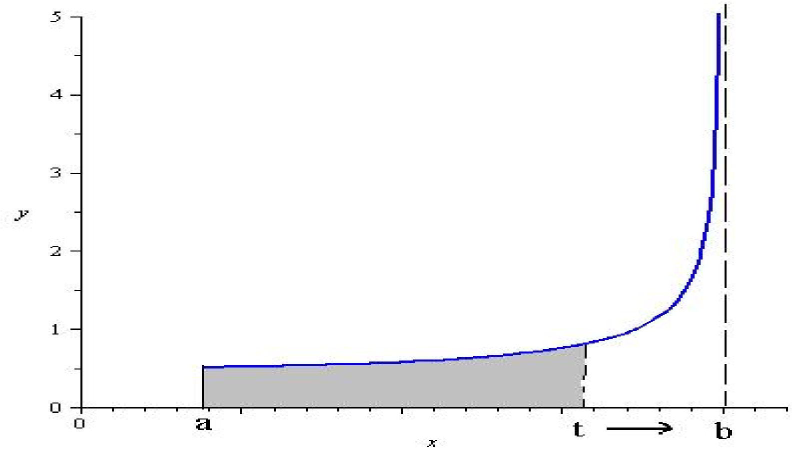


Figure: Area interpretation of improper integral where f has a vertical asymptote at b.

Evaluate the Improper Integral if Possible

$f(x) = \frac{1}{\sqrt{x}}$ which isn't defined @ 0

(a) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} \int_t^1 x^{-1/2}\,dx$

$$= \lim_{t\to 0^+} \left. \frac{x^{1/2}}{1/2} \right|_t^1$$

$$= \lim_{t\to 0^+} \left. 2\sqrt{x} \right|_t^1$$

$$= \lim_{t\to 0^+} \left(2\sqrt{1} - 2\sqrt{t}\right) = 2 - 0 = 2$$

The integral is convergent with value 2.

Evaluate the Improper Integral if Possible

$f(x) = \frac{1}{x-2}$ which is undefined @ 2

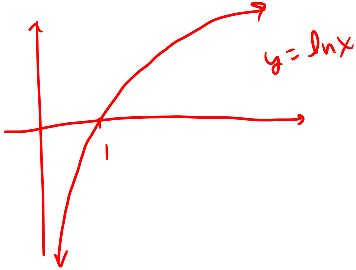
(b) $\displaystyle\int_1^2 \frac{dx}{x-2}$

$$= \lim_{t \to 2^-} \int_1^t \frac{dx}{x-2}$$

$$= \lim_{t \to 2^-} \ln|x-2| \Big|_1^t$$

$$= \lim_{t \to 2^-} \left(\ln|t-2| - \ln|1-2| \right) = -\infty - \ln 1 = -\infty$$

The integral is divergent.



$$\lim_{x \to 0^+} \ln x = -\infty$$

Evaluate the Improper Integral if Possible

(c) $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

$f(x) = \dfrac{1}{\sqrt{9-x^2}}$

is not defined @ 3

$$= \lim_{t \to 3^-} \int_0^t \frac{dx}{\sqrt{9-x^2}}$$

$$= \lim_{t \to 3^-} \sin^{-1}\left(\frac{x}{3}\right)\Big|_0^t$$

$$= \lim_{t \to 3^-} \left(\sin^{-1}\left(\frac{t}{3}\right) - \sin^{-1}\left(\frac{0}{3}\right)\right) = \sin^{-1}(1) - \sin^{-1}(0)$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

The integral is convergent.

Evaluate the Improper Integral if Possible

(a) $\int_0^1 \ln x\,dx = \lim_{t\to 0^+} \int_t^1 \ln x \, dx$

$= \lim_{t\to 0^+} \left(x\ln x - x\right)\Big|_t^1$

$= \lim_{t\to 0^+} \left[1\ln 1 - 1 - \left(t\ln t - t\right)\right]$

$= \lim_{t\to 0^+} \left(-1 - t\ln t + t\right)$

$$= -1 - 0 + 0 \qquad = -1$$

The integral is convergent.

$*$ $\int \ln x \, dx$

Use Int by parts

$u = \ln x \qquad du = \frac{1}{x} dx$

$v = x \qquad dv = dx$

$= x \ln x - \int x \cdot \frac{1}{x} dx$

$= x \ln x - \int dx \quad = x \ln x - x + C$

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$$\lim_{t\to 0^+} t\ln t = \text{"}0\cdot(-\infty)\text{"}$$ indeterminate form

$$= \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}} = \text{"}\frac{-\infty}{\infty}\text{"}$$ use l'Hospital's rule

$$= \lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{-1}{t^2}} = \lim_{t\to 0^+} \frac{1}{t}(-t^2) = \lim_{t\to 0} -t = 0$$

Example

Show that the integral is divergent

$$\int_{-2}^{1} \frac{1}{x^2}\, dx$$

$$\int_{-2}^{1} \frac{dx}{x^2} = \int_{-2}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2}$$

$$\int_{-2}^{0} \frac{dx}{x^2} = \lim_{t \to 0^-} \int_{-2}^{t} x^{-2}\, dx$$

$$= \lim_{t \to 0^-} \left. \frac{x^{-1}}{-1} \right|_{-2}^{t}$$

$$= \lim_{t \to 0^-} \left(\frac{-1}{t} - \frac{-1}{-2} \right) = \infty$$

This diverges, hence

$$\int_{-2}^{1} \frac{dx}{x^2} \quad \text{diverges.}$$