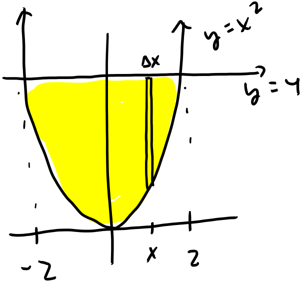
June 2 Math 2254 sec 001 Summer 2015

Section 6.4: Volume of a Solid by Slicing

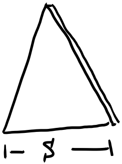
Volume: Let S be a solid that that lies between $x = a$ and $x = b$ having cross sectional area $A(x)$, where the cross section is in the plane through the solid perpendicular to the x-axis at each x in (a, b). The volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x = \int_a^b A(x)\, dx.$$

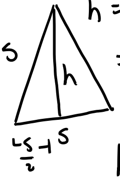
An object has as its base the region bounded between $y = x^2$ and $y = 4$. Cross sections taken perpendicular to the x-axis are equilateral triangles with one side in the plane. Find the volume of the solid.



$x^2 = 4 \Rightarrow x = -2$ or $x = 2$



$s = 4 - x^2$ (Top - bottom)



$h^2 = s^2 - \left(\frac{s}{2}\right)^2$

$= \frac{3}{4} s^2$

$h = \frac{\sqrt{3}}{2} s$

$A = \frac{1}{2} hb = \frac{1}{2} \frac{\sqrt{3}}{2} s \cdot s$

$= \frac{\sqrt{3}}{4} s^2$

Volume by Cross Section Applet 2

$$A(x) = \frac{\sqrt{3}}{4}(4-x^2)^2$$

$$V = \int_{-2}^{2} \frac{\sqrt{3}}{4}(4-x^2)^2\,dx$$ using symmetry

$$= 2\int_{0}^{2} \frac{\sqrt{3}}{4}(4-x^2)^2\,dx$$

$$= \frac{\sqrt{3}}{2}\int_{0}^{2}(16-8x^2+x^4)\,dx$$

$$= \frac{\sqrt{3}}{2}\left[16x - 8\frac{x^3}{3} + \frac{x^5}{5}\right|_{0}^{2}$$

$$= \frac{\sqrt{3}}{2}\left[16\cdot 2 - \frac{8}{3}(2)^3 + \frac{2^5}{5}\right] - 0$$

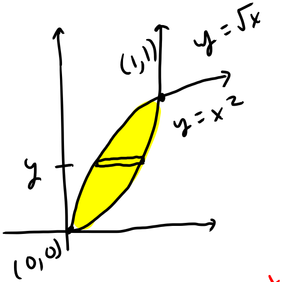
$$= \frac{\sqrt{3}}{2}\left[32 - \frac{64}{3} + \frac{32}{5}\right]$$

$$= \frac{\sqrt{3}}{2}(32)\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$= 16\sqrt{3}\left(\frac{1}{3} + \frac{1}{5}\right) = 16\sqrt{3}\left(\frac{5+3}{15}\right)$$

$$= \frac{128\sqrt{3}}{15}$$

An object has as its base the region bounded between $y = \sqrt{x}$ and $y = x^2$. Cross sections taken perpendicular to the y-axis are squares with one side in the plane. Find the volume of the solid.

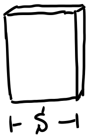


intersections $\sqrt{x} = x^2 \Rightarrow x = 0$

$x = x^4 \Rightarrow 0 = x^4 - x$

$0 = x(x^3 - 1)$

or $x = 1$



$A = s^2$

$y = x^2 \Rightarrow x = \sqrt{y}$ right

$y = \sqrt{x} \Rightarrow x = y^2$ left

$s = \sqrt{y} - y^2$

$A(y) = (\sqrt{y} - y^2)^2$

$$V = \int_0^1 (\sqrt{y} - y^2)^2 \, dy$$

$$= \int_0^1 \left((\sqrt{y})^2 - 2\sqrt{y}\, y^2 + y^4 \right) dy$$

$$= \int_0^1 \left(y - 2\, y^{5/2} + y^4 \right) dy$$

$$= \frac{y^2}{2} - 2\, \frac{y^{7/2}}{7/2} + \frac{y^5}{5} \Big|_0^1$$

$$= \frac{y^2}{2} - \frac{4}{7} y^{7/2} + \frac{y^5}{5} \Big|_0^1$$

$$= \frac{1^2}{2} - \frac{4}{7}(1)^{7/2} + \frac{1^5}{5} - 0$$

$$= \frac{1}{2} - \frac{4}{7} + \frac{1}{5}$$

$$= \frac{35}{70} - \frac{40}{70} + \frac{14}{70} = \frac{9}{70}$$

Section 6.2: Solid of Revolution, Disks and Washers

Consider a region bounded under the nonnegative function $y = f(x)$ for $a \le x \le b$. If this region is rotated about the x-axis, a solid is formed. The cross sections of this solid will be circles with radius $f(x)$. So the volume of such a solid is

$$\text{Solid of Revolution:} \quad V = \int_a^b \pi (f(x))^2\, dx$$

This is called the method of **disks.** Each very thin slice is a disk.

Note: Because cross sections are circles, the area function

$$A(x) = \pi (radius)^2 = \pi (f(x))^2$$

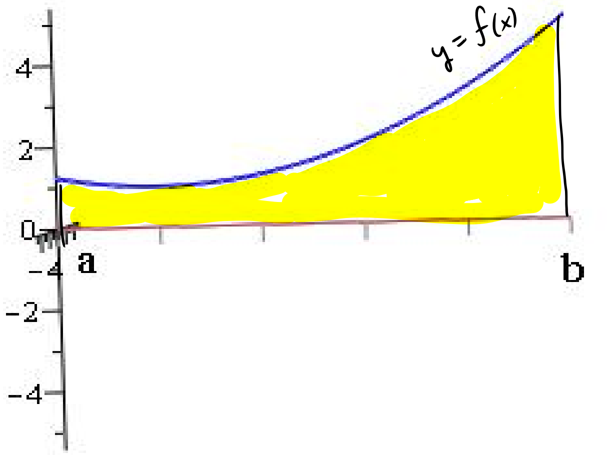


Figure: Start with a positive function $y = f(x)$ and the region below the curve on $[a, b]$

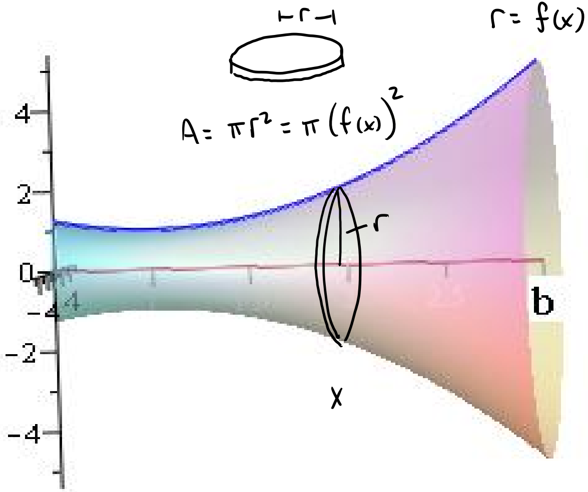


Figure: Revolve it about the *x*-axis to get a solid whose cross sections are circular disks.

Derive the formula for the volume of a cone $V = \frac{\pi}{3} r^2 h$.

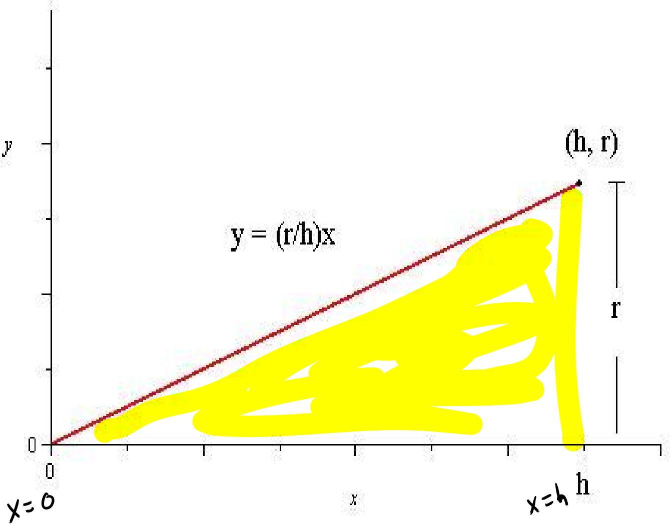


Figure: Start with the line $y = \frac{r}{h}x$ for $0 \le x \le h$.

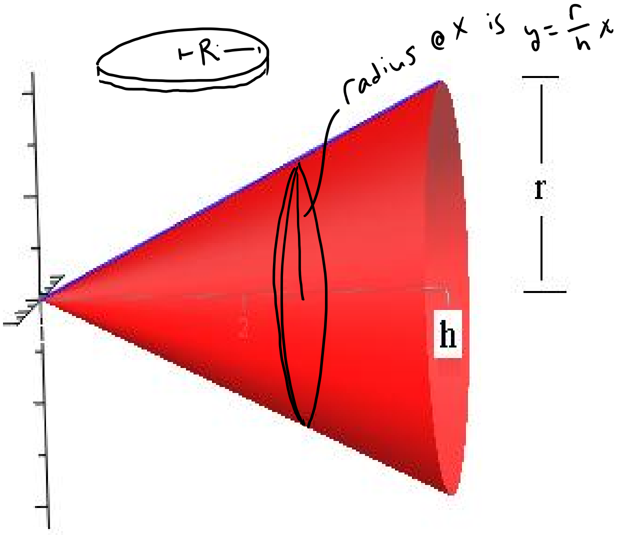


Figure: And revolve this line about the x-axis to get the cone.

$a = 0$ and $b = h$ $\qquad f(x) = \frac{r}{h}x$

$$V = \int_a^b \pi \left(f(x)\right)^2 dx = \int_0^h \pi \left(\frac{r}{h}x\right)^2 dx$$

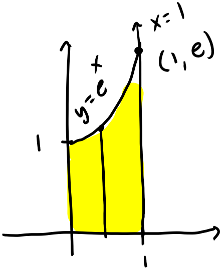
$$= \int_0^h \pi \frac{r^2}{h^2} x^2 dx$$

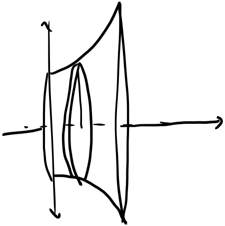
$$= \frac{\pi r^2}{h^2} \int_0^h x^2 dx = \frac{\pi r^2}{h^2} \left(\frac{x^3}{3}\Big|_0^h\right) = \frac{\pi r^2}{h^2}\left(\frac{h^3}{3} - 0\right)$$

$$= \pi r^2 \frac{h}{3} = \frac{\pi}{3} r^2 h$$

Example

The first quadrant region bounded by the curve $y = e^x$, the y-axis, and the line $x = 1$ is rotated about the x-axis. Find the volume of the resulting solid.





$r = e^x$

$$V = \int_0^1 \pi \left(e^x\right)^2 dx$$

$* \quad \left(e^x\right)^2 = e^{2x}$

$$V = \int_0^1 \pi e^{2x}\, dx$$

$$= \pi \cdot \frac{1}{2} e^{2x} \Big|_0^1$$

$$= \frac{\pi}{2}\left[e^2 - e^0\right] = \frac{\pi}{2}\left(e^2 - 1\right)$$

Washers: (solid with a solid part removed)

Suppose we consider the region bounded between two curves $y = f(x)$ and $y = g(x)$ for $a \le x \le b$ with $0 \le g(x) \le f(x)$. If this region is rotated about the x-axis, a solid is generated with volume

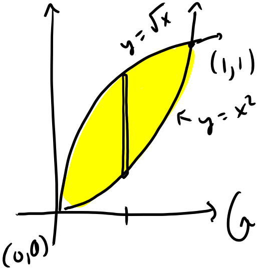
$$V = \int_a^b \pi((f(x))^2 - (g(x))^2)\, dx$$

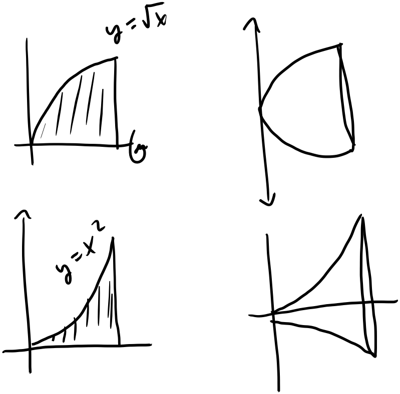
This is called the method of **washers**. Each very thin slice is shaped like a washer (a disk with a concentric disk removed).

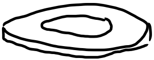
Note that the above formula is equivalent to

$$V = \left[\int_a^b \pi((f(x))^2\, dx\right] - \left[\int_a^b \pi(g(x))^2)\, dx\right]$$

Find the volume of the solid obtained by rotating the region bounded between $y = \sqrt{x}$ and $y = x^2$ about the x-axis.







Outer radius $f(x) = \sqrt{x}$

inner radius $g(x) = x^2$

Volume by Washers Applet

$$V = \int_0^1 \pi\left((\sqrt{x})^2 - (x^2)^2\right)dx$$

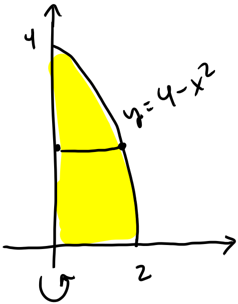
$$= \int_0^1 \pi\,(x - x^4)\,dx$$

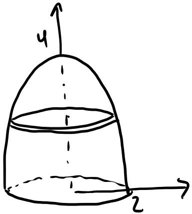
$$= \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right|_0^1$$

$$= \pi\left[\frac{1^2}{2} - \frac{1^2}{5} - 0\right] = \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \pi\left(\frac{5-2}{10}\right)$$

$$= \frac{3\pi}{10}$$

The first quadrant region bounded between $y = 4 - x^2$ and the x and y axes is rotated about the y**-axis**. Find the volume of the resulting solid.





Cross section perp. to y-axis

$r = \sqrt{4-y}$

$y = 4 - x^2$

$x^2 = 4 - y$

$x = \sqrt{4-y}$

top y

$$V = \int_0^4 \pi \left(\sqrt{4-y}\right)^2 dy$$

bottom y

$$V = \pi \int_0^4 (4-y)\,dy$$

$$= \pi \left[4y - \frac{y^2}{2}\right|_0^4$$

$$= \pi \left[4\cdot 4 - \frac{4^2}{2} - 0\right]$$

$$= \pi\,[16 - 8] = 8\pi$$