June 30 Math 2254 sec 001 Summer 2015

Section Section 7.8: Improper Integrals

Comparison Theorem for Improper Integrals

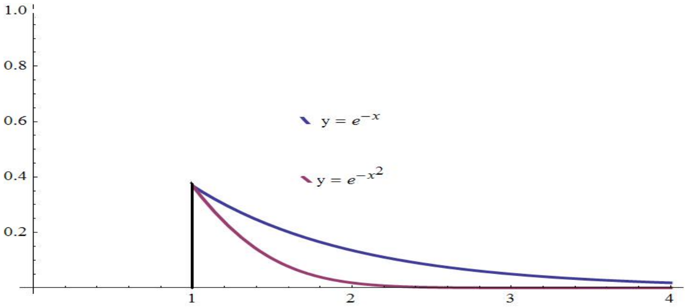

Figure: Graphs of $y = e^{-x}$ and $y = e^{-x^2}$ together. Note that for all $x \geq 1$, $e^{-x^2} < e^{-x}$. What should be true about $\int_1^\infty e^{-x}\,dx$ and $\int_1^\infty e^{-x^2}\,dx$?

Comparison

Evaluate the improper integral $\displaystyle\int_1^{\infty} e^{-x}\,dx$

$$\int_1^{\infty} e^{-x}\,dx = \lim_{t\to\infty} \int_1^{t} e^{-x}\,dx$$

$$= \lim_{t\to\infty} -e^{-x}\Big|_1^{t}$$

$$= \lim_{t\to\infty} \left[-e^{-t} - (-e^{-1})\right] = 0 + e^{-1} = \frac{1}{e}$$

The integral is convergent.

Comparison

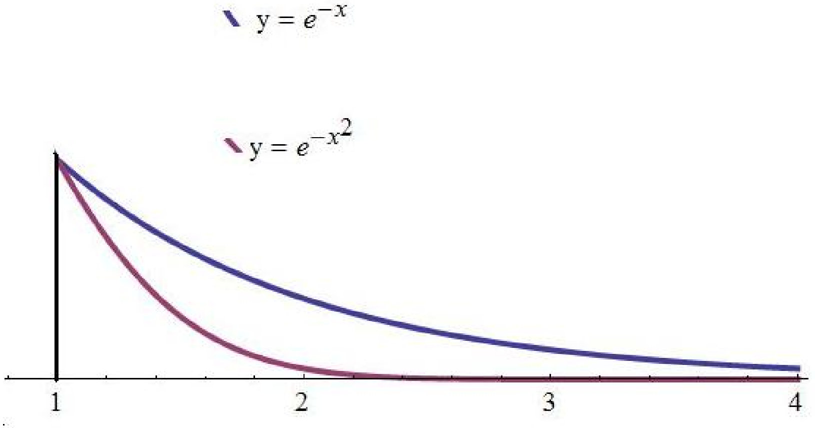

Figure: If the area under the blue curve is finite, can the area under the red curve be infinite?

Comparison Theorem

Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then

(a) if $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.

(b) if $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

Example

Determine if the integral

$$\int_1^\infty e^{-x^2}\,dx$$

is convergent or divergent.

For all $x \geqslant 1$ $\quad e^{-x} \geqslant e^{-x^2} \geqslant 0$

and $\int_1^\infty e^{-x}\,dx$ is convergent.

Hence $\int_1^\infty e^{-x^2}\,dx$ converges by the comparison theorem.

Here, $f(x) = e^{-x}$ and $g(x) = e^{-x^2}$

Recall $\int_1^\infty \frac{1}{x}\,dx$ diverges.

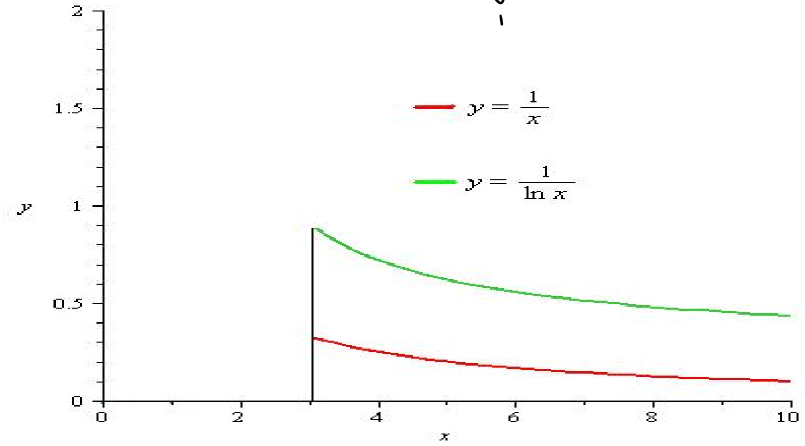

Figure: The curves $y = \frac{1}{x}$ and $y = \frac{1}{\ln x}$ plotted together for $x \geq 3$.

Example

Determine if the integral

$$\int_3^\infty \frac{dx}{\ln x}$$

is convergent or divergent.

For all $x \geqslant 3$ $\quad \frac{1}{\ln x} \geqslant \frac{1}{x} \geqslant 0$

and $\int_3^\infty \frac{1}{x}\,dx$ diverges.

Hence $\int_3^\infty \frac{1}{\ln x}\,dx$ diverges by the comparison theorem.

Here $f(x) = \frac{1}{\ln x}$ and $g(x) = \frac{1}{x}$

Determine if the Integral Converges or Diverges

(a) $$\int_{\pi}^{\infty} \frac{2+\sin x}{x^3}\,dx$$

Recall $\int_1^{\infty} \frac{1}{x^3}\,dx$ converges.

$p = 3 > 1$

$1 \le 2+\sin x \le 3$ since $-1 \le \sin x \le 1$

so $\frac{1}{x^3} \le \frac{2+\sin x}{x^3} \le \frac{3}{x^3}$

Since $\frac{3}{x^3} \ge \frac{2+\sin x}{x^3} \ge 0$ for $x \ge \pi$

and $\int_{\pi}^{\infty} \frac{3}{x^3}\,dx$ converges, $\int_{\pi}^{\infty} \frac{2+\sin x}{x^3}\,dx$

converges by the comparison theorem.

Determine if the Integral Converges or Diverges

(b) $\displaystyle\int_1^{\infty} \frac{\arctan x}{\sqrt{x}}\, dx$

Recall $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x}}$ diverges

$p = \frac{1}{2} \leq 1$

For $x \geq 1$ $\quad \frac{\pi}{4} \leq \arctan x \leq \frac{\pi}{2}$

so $\displaystyle\frac{\frac{\pi}{4}}{\sqrt{x}} \leq \frac{\arctan x}{\sqrt{x}} \leq \frac{\pi/2}{\sqrt{x}}$

Since $\displaystyle\frac{\arctan x}{\sqrt{x}} \geq \frac{\pi/4}{\sqrt{x}} \geq 0$ for $x \geq 1$

and $\displaystyle\int_1^{\infty} \frac{\pi/4}{\sqrt{x}}\, dx$ diverges, $\displaystyle\int_1^{\infty} \frac{\arctan x}{\sqrt{x}}\, dx$

diverges by the comparison theorem.

Section 8.1: Sequences

Definition: A **sequence** is a function whose domain is a subset of the integers and whose range is a subset of the real numbers.

Typically, the domain is the positive $\{1, 2, 3, \ldots\}$ or the nonnegative $\{0, 1, 2, \ldots\}$ integers.

A sequence is often presented as a list of numbers $f(1), f(2), f(3), \ldots$ (think *comma separated list*).

Here, $f(1)$ is called the *first term*, $f(2)$ is called the *second term*, and in general

$$f(n) \quad \text{is called the } n^{th} \text{ term in the sequence.}$$

Defining Sequences and Notation

Example: Consider the function $f(n) = e^{-n}$ with domain $\{0, 1, 2, \ldots\}$.

We can represent this as a list $1, e^{-1}, e^{-2}, \ldots$

We can represent this by giving it a short hand name and using a subscript $s_n = f(n)$

$$s_0 = f(0) = 1, \quad s_1 = f(1) = e^{-1}, \quad s_2 = f(2) = e^{-2}, \ldots$$

We can also represent the sequence using curly bracket notation

$$\{s_n\} = \left\{e^{-n}\right\}, \quad \text{or to highlight the domain} \quad \{s_n\}_{n=0}^{\infty} = \left\{e^{-n}\right\}_{n=0}^{\infty}$$

Plot of a Sequence

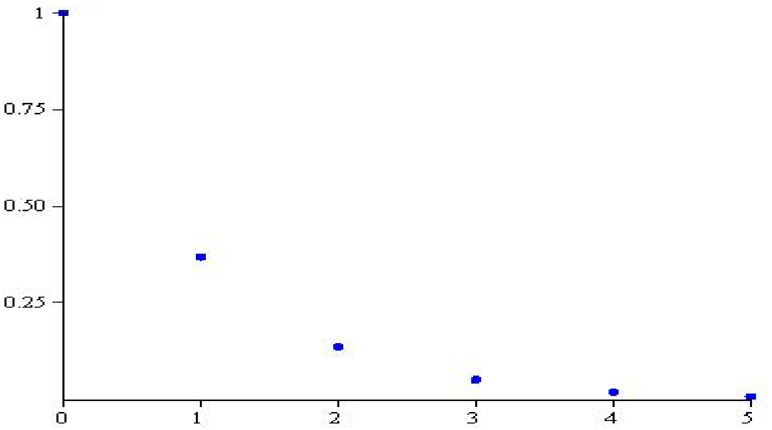

Figure: Plot of the sequence $\{e^{-n}\}$. Note that the graph consists of distinct **points**.

Examples

Write the first four terms of the sequence defined by the indicated relation.

(a) $\{a_n\}_{n=1}^{\infty} = \left\{ \dfrac{2n}{n+1} \right\}_{n=1}^{\infty}$

$a_1 = \frac{2\cdot1}{1+1} = \frac{2}{2} = 1$

$a_2 = \frac{2\cdot2}{2+1} = \frac{4}{3}$

$a_3 = \frac{2\cdot3}{3+1} = \frac{6}{4} = \frac{3}{2}$

$a_4 = \frac{2\cdot4}{4+1} = \frac{8}{5}$

(b) $\{a_n\}_{n=0}^{\infty} = \{(-1)^n\}_{n=0}^{\infty}$

$a_0 = (-1)^0 = 1$, $a_1 = (-1)^1 = -1$, $a_2 = (-1)^2 = 1$, $a_3 = (-1)^3 = -1$

A recursively defined sequence

The **Fibonacci sequence**, $\{f_n\}$ is defined by

$$f_0 = 1, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad \text{for} \quad n \geq 2.$$

Write the first 6 terms of this sequence.

$f_0 = 1$

$f_1 = 1$

$f_2 = f_1 + f_0 = 1 + 1 = 2$

$f_3 = f_2 + f_1 = 2 + 1 = 3$

$f_4 = f_3 + f_2 = 3 + 2 = 5$

$f_5 = f_4 + f_3 = 5 + 3 = 8$