June 5 Math 1190 sec. 51 Summer 2017

Section 1.2: Limits of Functions Using Properties of Limits

We had the following results:

Theorem: If $f(x) = A$ where A is a constant, then for any real number c

$$\lim_{x\to c} f(x) = \lim_{x\to c} A = A$$

Theorem: If $f(x) = x$, then for any real number c

$$\lim_{x\to c} f(x) = \lim_{x\to c} x = c$$

Additional Limit Law Theorems

Suppose

$$\lim_{x\to c} f(x) = L, \quad \lim_{x\to c} g(x) = M, \quad \text{and } k \text{ is constant.}$$

Theorem: (Sums) $\lim_{x\to c}(f(x)+g(x)) = L+M$

Theorem: (Differences) $\lim_{x\to c}(f(x)-g(x)) = L-M$

Theorem: (Constant Multiples) $\lim_{x\to c} kf(x) = kL$

Theorem: (Products) $\lim_{x\to c} f(x)g(x) = LM$

Additional Limit Law Theorems

For positive integer n

Theorem: (Power) $\lim_{x\to c}(f(x))^n = L^n$

Note in particular that this tells us that $\lim_{x\to c} x^n = c^n$.

Theorem: (Root) $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$ (if this is defined)

Theorem: (Quotient) $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, if $M \neq 0$

Theorem: If $R(x)$ is rational, and c is in the domain of R, then

$$\lim_{x\to c} R(x) = R(c).$$

Example: Using the laws directly

Evaluate if possible $\lim_{t\to 3} \dfrac{t^2-9}{t+2}$

$\dfrac{t^2-9}{t+2}$ is rational with 3 in its domain since $3+2\neq 0$

so

$$\lim_{t\to 3} \frac{t^2-9}{t+2} = \frac{3^2-9}{3+2} = \frac{0}{5} = 0$$

Additional Techniques: When direct laws fail

Evaluate if possible $\lim_{x \to -1} \dfrac{x+1}{x^3+1}$

-1 is not in the domain of $\dfrac{x+1}{x^3+1}$ because $(-1)^3+1 = -1+1 = 0$

Note that when $x=-1$, $x+1 = -1+1 = 0$ too. The ratio looks like $\frac{0}{0}$ as x goes to -1.

Since -1 is a zero of x^3+1, $x-(-1) = x+1$ is a factor of x^3+1.

Recall the sum of cubes formula $a^3+b^3 = (a+b)(a^2-ab+b^2)$

So $x^3+1 = (x+1)(x^2-1\cdot x+1^2)$ here $a=x$ and $b=1$

Hence $\displaystyle\lim_{x\to -1} \frac{x+1}{x^3+1} = \lim_{x\to -1} \frac{x+1}{(x+1)(x^2-x+1)} =$

Cancel the common factor of $x+1$

$$= \lim_{x\to -1} \frac{\cancel{x+1}}{\cancel{(x+1)}(x^2-x+1)} = \lim_{x\to -1} \frac{1}{x^2-x+1}$$

$$= \frac{1}{(-1)^2-(-1)+1} = \frac{1}{3}$$

Additional Techniques: When direct laws fail

Evaluate if possible $\displaystyle\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1}$

We can't plug in 1 since $x-1$ becomes zero.

Note, the numerator $\sqrt{x+3}-2$ also tends to

$\sqrt{1+3}-2 = \sqrt{4}-2 = 2-2 = 0$.

We see $\frac{0}{0}$ again. We can use the conjugate to get the radical out of the numerator.

The conjugate of $\sqrt{x+3}-2$ is $\sqrt{x+3}+2$

We'll take the limit by multiplying by 1 in the form

$1 = \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$.

$$\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} = \lim_{x\to 1} \left(\frac{\sqrt{x+3}-2}{x-1}\right)\cdot\left(\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\right)$$

$$= \lim_{x\to 1} \frac{\left(\sqrt{x+3}\right)^2 - 2\sqrt{x+3} + 2\sqrt{x+3} - 4}{(x-1)\left(\sqrt{x+3}+2\right)}$$

$$= \lim_{x\to 1} \frac{x+3-4}{(x-1)\left(\sqrt{x+3}+2\right)} = \lim_{x\to 1} \frac{x-1}{(x-1)\left(\sqrt{x+3}+2\right)}$$

Cancel common factor of $x-1$

$$= \lim_{x\to 1} \frac{\cancel{x-1}}{\cancel{(x-1)}\left(\sqrt{x+3}+2\right)} = \lim_{x\to 1} \frac{1}{\sqrt{x+3}+2} = \frac{1}{\sqrt{1+3}+2}$$

$$= \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

Question

Evaluate if possible $\lim_{x\to 2} \frac{x-2}{\sqrt{x}-\sqrt{2}}$ $= \lim_{x\to 2} \left(\frac{x-2}{\sqrt{x}-\sqrt{2}}\right)\cdot\left(\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}\right)$

$= \lim_{x\to 2} \frac{(\cancel{x-2})(\sqrt{x}+\sqrt{2})}{\cancel{x-2}}$

(a) $\frac{1}{\sqrt{2}}$

(b) $\sqrt{2}$ $= \lim_{x\to 2} (\sqrt{x}+\sqrt{2}) = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$

(c) DNE

(d) $2\sqrt{2}$

$(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2}) = (\sqrt{x})^2 - \sqrt{2}\sqrt{x} + \sqrt{2}\sqrt{x} - (\sqrt{2})^2$

Observations

In limit taking, the form "$\frac{0}{0}$" sometimes appears. This is called an **indeterminate form.** Standard strategies are

(1) Try to factor the numerator and denominator to see if a common factor–$(x - c)$–can be cancelled.

(2) If dealing with roots, try rationalizing to reveal a common factor.

The form

$$\text{"}\frac{\text{nonzero constant}}{0}\text{"}$$

is not indeterminate. It is undefined. When it appears, the limit doesn't exist.

Example

Let $f(x) = x^3 + 2x$. Determine the difference quotient

$$\frac{f(x+h) - f(x)}{h} \quad \text{for} \quad h \neq 0.$$

Next, take the limit as $h \to 0$ of this difference quotient.

$$f(x) = x^3 + 2x$$

$$f(x+h) = (x+h)^3 + 2(x+h)$$

$$= x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h$$

$$\frac{f(x+h) - f(x)}{h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h - (x^3 + 2x)}{h}$$

$$= \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 + \cancel{2x} + 2h - \cancel{x^3} - \cancel{2x}}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3 + 2h}{h}$$

$$= \frac{\cancel{h}\left(3x^2 + 3xh + h^2 + 2\right)}{\cancel{h}}$$

$$= 3x^2 + 3xh + h^2 + 2$$

Now we'll take the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\left(3x^2+3xh+h^2+2\right)$$

$$= 3x^2+3x\cdot 0+0^2+2$$

$$= 3x^2+2$$

* we treat x like the constant 2 since the limit is a h goes to zero. *

Section 1.3: Continuity

We have seen that their may or may not be a relationship between the quantities

$$\lim_{x\to c} f(x) \quad \text{and} \quad f(c).$$

One or the other (or both) may fail to exist. And even if both exist, they need not be equivalent.

We've also seen that for polynomials at least, that the limit at a point is the same as the function value at that point. Here, we explore this property that polynomials (and lots of other functions, but not all) share.

Definition: Continuity at a Point

Definition: A function f is continuous at a number c if

$$\lim_{x \to c} f(x) = f(c).$$

Note that three properties are contained in this statement:
(1) $f(c)$ is defined (i.e. c is in the domain of f),

(2) $\lim_{x \to c} f(x)$ exists, and

(3) the limit actually equals the function value.

If a function f is not continuous at c, we may say that f is **discontinuous** at c

Polynomials and Rational Functions

In the previous section, we saw that:

If P is any polynomial and c is any real number, then $\lim_{x\to c} P(x) = P(c)$, and

If R is any rational function and c is any number in the domain of R, then $\lim_{x\to c} R(x) = R(c)$.

Conclusion Theorem: Every rational function[1] is continuous at each number in its domain.

[1]Note that polynomials can be lumped in to the set of all rational functions.

Examples: Determine where each function is discontinuous.

$$g(t) = \frac{t^2 - 9}{t + 3}$$

g is rational, hence continuous on its domain.

note
$t+3=0$ if $t=-3$

If $t=-3$, the denominator would be zero.

Since -3 is not in g's domain,

g is discontinuous @ -3.

$$f(x) = \begin{cases} 2x, & x < 1 \\ x^2+1, & 1 \le x < 2 \\ 3, & x \ge 2 \end{cases}$$

The pieces $y=2x$, $y=x^2+1$, and $y=3$ are polynomial. So they are each continuous where the hold.

f is continuous at each number c if $c<1$, $1<c<2$, and $c>2$.

What about $c=1$? ① Is $f(1)$ defined? Yes $f(1)=1^2+1=2$.

② Does $\lim_{x\to 1} f(x)$ exist?

$$\left.\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} 2x = 2(1) = 2 \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} (x^2+1) = 1^2+1 = 2 \end{aligned}\right\} \lim_{x\to 1} f(x) = 2$$

③ Does $\lim_{x\to 1} f(x) = f(1)$? Yes, $\lim_{x\to 1} f(x) = 2 = f(1)$.

f is continuous at 1.

What about @ $x=2$?

① Is $f(2)$ defined? Yes $f(2) = 3$

② Does $\lim_{x\to 2} f(x)$ exist?

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2+1) = 2^2+1 = 5$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 3 = 3$$

$\lim_{x\to 2} f(x)$ DNE!

f is not continuous at 2 since the limit doesn't exist.

Link to plot of f

Question

Determine whether f is continuous at 1 where $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

$f(1) = 2$

(a) No because $f(1)$ is not defined.

(b) Yes because all three conditions hold.

(c) No because $\lim_{x\to 1} f(x)$ doesn't exist.

(d) No because f is piecewise defined.

$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1}$

$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}$

$= \lim_{x\to 1} (x+1) = 1+1 = 2$

$= f(1)$

Removable and Jump Discontinuities

Definition: Let f be defined on an open interval containing c except possibly at c. If $\lim_{x\to c} f(x)$ exists, but f is discontinuous at c, then f has a **removable discontinuity** at c.

this is a hole in the graph

Definition: If $\lim_{x\to c^-} f(x) = L_1$ and $\lim_{x\to c^+} f(x) = L_2$ where $L_1 \neq L_2$ (i.e. both one sided limits exist but are different), then f has a **jump discontinuity** at c.

Removable and Jump Discontinuities

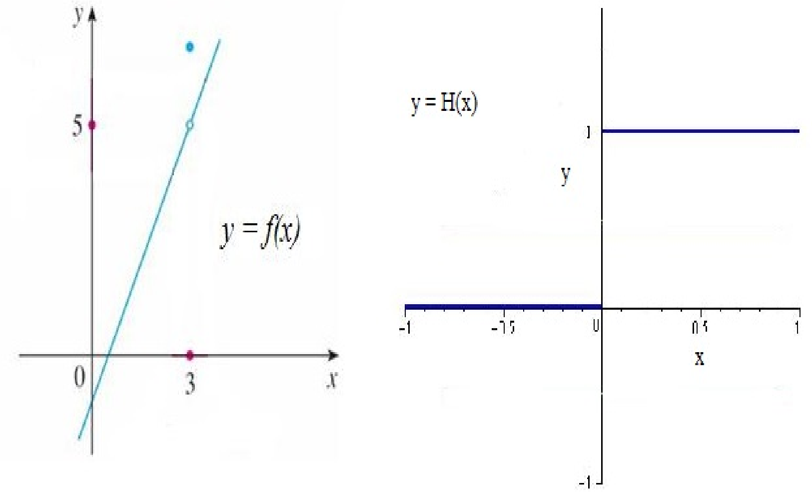


Figure: Example of a removable (left) discontinuity and a jump (right) discontinuity.

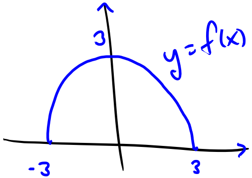
One Sided Continuity Example:

Consider the function $f(x) = \sqrt{9 - x^2}$. Plot a rough sketch of the graph of f, and determine its domain.

If $y = \sqrt{9 - x^2}$, then $y^2 = 9 - x^2 \Rightarrow x^2 + y^2 = 9 = 3^2$

circle centered @ $(0,0)$ radius 3

$f(x) = \sqrt{9 - x^2}$ is the positive root of $9 - x^2$, hence the top half of the circle.



$\boxed{\text{Domain of } f \text{ is } -3 \le x \le 3}$

Note $9 - x^2 \ge 0 \Rightarrow 9 \ge x^2$

$\Rightarrow \sqrt{9} \ge \sqrt{x^2} = |x|$

$\Rightarrow -3 \le x \le 3$

$f(x) = \sqrt{9 - x^2}$

Note that f is continuous on $-3 < x < 3$. What can be said about

$$\lim_{x\to -3} f(x) \qquad \text{or} \qquad \lim_{x\to 3} f(x)?$$

These limits don't exist. Since f is not defined if $x > 3$, $\lim_{x\to 3^+} f(x)$ DNE.

Similarly, $\lim_{x\to -3^-} f(x)$ DNE.

Continuity From the Left & Right

Definition: Let a function f be defined on an interval $[c, b)$. Then f is continuous from the right at c if

$$\lim_{x \to c^+} f(x) = f(c).$$

Let f be defined on an interval $(a, c]$. Then f is continuous from the left at c if

$$\lim_{x \to c^-} f(x) = f(c).$$

Example: $f(x) = \sqrt{9 - x^2}$

Show that f is continuous from the right at -3.

f is defined on $[-3, 3]$. $f(-3) = \sqrt{9 - (-3)^2} = \sqrt{9-9} = 0$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9 - x^2} = \sqrt{9 - (-3)^2} = 0$$

Note $\lim_{x \to -3^+} f(x) = f(-3)$.

So f is continuous from the right @ -3.

A Theorem on Continuous Functions

Theorem If f and g are continuous at c and for any constant k, the following are also continuous at c:

$$(i)\, f+g, \quad (ii)\, f-g, \quad (iii)\, kf, \quad (iv)\, fg, \quad \text{and} \quad (v)\, \frac{f}{g}, \text{ if } g(c)\neq 0.$$

In other words, if we combine continuous functions using addition, subtraction, multiplication, division, and using constant factors, the result is also continuous—provided of course that we don't introduce division by zero.

Questions

(1) **True or False** If f is continuous at 3 and g is continuous at 3, then it must be that

$$\lim_{x\to 3} f(x)g(x) = f(3)g(3).$$

product of cont functions is cont.

(2) **True or False** If $f(2) = 1$ and $g(2) = 7$, then it must be that

Consider the example

$$\lim_{x\to 2} \frac{f(x)}{g(x)} = \frac{1}{7}.$$

$f(x) = 1$, $\quad g(x) = \begin{cases} \frac{1}{x-2}, & x \neq 2 \\ 7, & x = 2 \end{cases}$.

Continuity on an Interval

Definition A function is continuous on an interval (a, b) if it is continuous at each point in (a, b). A function is continuous on an interval such as $(a, b]$ or $[a, b)$ or $[a, b]$ provided it is continuous on (a, b) and has one sided continuity at each included end point.

Graphically speaking, if $f(x)$ is continuous on an interval (a, b), then the curve $y = f(x)$ will have no holes or gaps.

Find all values of A such that f is continuous on $(-\infty, \infty)$.

f is continuous at all $x \neq 2$.

$$f(x) = \begin{cases} x + A, & x < 2 \\ Ax^2 - 3, & 2 \leq x \end{cases}$$

Note that $f(2) = A(2)^2 - 3 = 4A - 3$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x+A) = 2+A$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (Ax^2-3) = 4A-3$$

If f is continuous at 2, then

$$2+A = 4A-3$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$

$$\Rightarrow \quad A = 4A-3-2 \Rightarrow A-4A = -5$$

$$-3A = -5 \Rightarrow A = \frac{5}{3}$$

$$\boxed{\text{So } f \text{ is continuous on } (-\infty,\infty) \text{ if } A = \tfrac{5}{3}.}$$

Compositions

Suppose $\lim_{x\to c} g(x) = L,$ and f is continuous at L, then

$$\lim_{x\to c} f(g(x)) = f(L) \quad \text{i.e.} \quad \lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right).$$

Theorem: If g is continuous at c and f is continuous at $g(c)$, then $(f\circ g)(x)$ is continuous at c.

Essentially, this says that "compositions of continuous functions are continuous."

Example

Suppose we know that $f(x) = e^x$ is continuous on $(-\infty, \infty)$[2]. Evaluate

$$\lim_{x\to\sqrt{\ln(3)}} e^{x^2+\ln(2)}$$

If $g(x) = x^2 + \ln 2$, it's continuous everywhere as a polynomial.

$$e^{x^2+\ln 2} = f(g(x)).$$

By continuity

$$\lim_{x\to\sqrt{\ln 3}} e^{x^2+\ln 2} = e^{(\sqrt{\ln 3})^2 + \ln 2}$$

$$= e^{\ln 3 + \ln 2} = e^{\ln 3} \cdot e^{\ln 2} = 3 \cdot 2 = 6$$

[2]This is true.

Inverse Functions

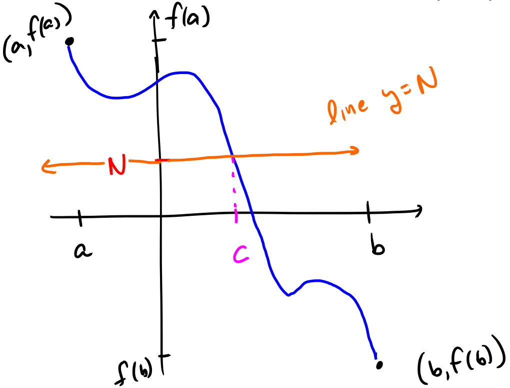
Theorem: If f is a one to one function that is continuous on its domain, then its inverse function f^{-1} is continuous on its domain.

Continuous functions (with inverses) have continuous inverses.

For Example: If we know that $\sin x$ is continuous on its domain, then we can conclude that $\sin^{-1} x$ is continuous on its domain.

Theorem:

Intermediate Value Theorem (IVT) Suppose f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$. Then there exists c in the interval (a, b) such that $f(c) = N$.



The theorem says that the line $y=N$ must intersect the graph of f.

c is the x-value where it happens.

The theorem doesn't tell us what c is, or that there is only one c. It does say there is at least one c.

Application of the IVT

Show that the equation has at least one solution in the interval.

$$x - 1 = 4 - x^3 \quad 1 \leq x \leq 2$$

Let $f(x) = x - 1 - 4 + x^3$. Note that if c satisfies $f(c) = 0$, then

$$c - 1 - 4 + c^3 = 0$$

$$\Rightarrow \quad c - 1 = 4 - c^3$$

That is, c would solve my original equation.

f is a polynomial, so f is continuous on $[1, 2]$.

And $f(1) = 1 - 1 - 4 + 1^3 = -3$, $f(2) = 2 - 1 - 4 + 2^3 = 5$

Note that $N=0$ is a number between $f(1)=-3$ and $f(2)=5$. By the IVT, there must be at least one number c between 1 and 2 such that $f(c)=0$. That is,

$x-1=4-x^3$ has at least one solution on the interval $(1,2)$.

Section 1.4: Limits and Continuity of Trigonometric, Exponential and Logarithmic Functions

Here we list without proof[3] the continuity properties of several well known functions.

$\sin x$: The sine function $y = \sin x$ is continuous on its domain $(-\infty, \infty)$.

$\cos x$: The cosine function $y = \cos x$ is continuous on its domain $(-\infty, \infty)$.

e^x: The exponential function $y = e^x$ is continuous on its domain $(-\infty, \infty)$.

$\ln(x)$: The natural log function $y = \ln(x)$ is continuous on its domain $(0, \infty)$.

[3] You are already familiar with their graphs.

Additional Functions

- By the quotient property, each of $\tan x$, $\cot x$, $\sec x$ and $\csc x$ are continuous on each of their respective domains.

- For $a > 0$ with $a \neq 1$, the function
$$a^x = e^{x \ln a}.$$
By the composition property, each exponential function $y = a^x$ is continuous on $(-\infty, \infty)$.

- For $a > 0$ with $a \neq 1$, the function
$$\log_a(x) = \frac{\ln x}{\ln a}.$$
By the constant multiple property, each logarithm function $y = \log_a(x)$ is continuous on $(0, \infty)$.

Example

Evaluate each limit.

(a) $\lim_{x\to\pi} \cos(x+\sin x)$

composition of continuous functions

$= \cos(\pi + \sin\pi)$

$= \cos(\pi + 0) = \cos(\pi) = -1$

(b) $\lim_{t\to\frac{\pi}{4}} e^{\tan t} = e^{\tan\frac{\pi}{4}} = e^{1} = e$

Question

Evaluate the limit $\lim_{x \to \pi} \ln(\cos^2 x)$. $= \ln(\cos^2 \pi)$

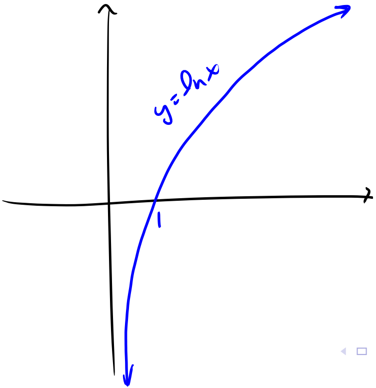
$= \ln((-1)^2) = \ln 1 = 0$

(a) e

(b) 1

(c) DNE

(d) 0



Squeeze Theorem:

Theorem: Suppose $f(x) \leq g(x) \leq h(x)$ for all x in an interval containing c except possibly at c. If

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$$

then

$$\lim_{x\to c} g(x) = L.$$

Squeeze Theorem:

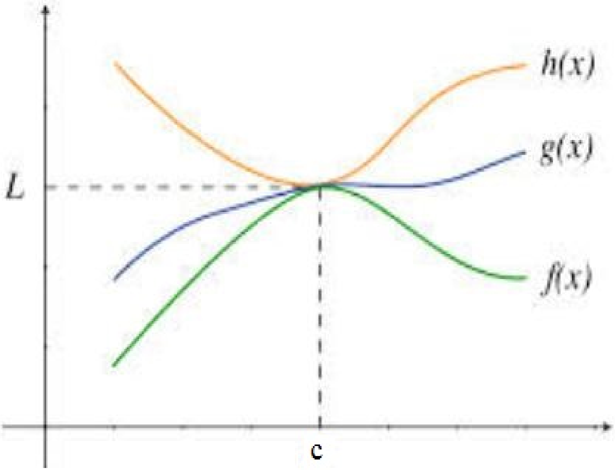


Figure: Graphical Representation of the Squeeze Theorem.