June 5 Math 2254 sec 001 Summer 2015

Section 6.3: Method of Cylindrical Shells

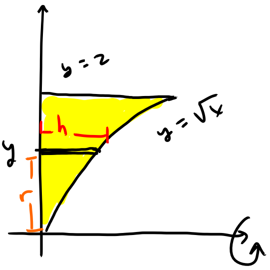
If the region bounded by the x-axis and the positive function $y = f(x)$ for $a \leq x \leq b$ is rotated about the y-axis, the volume V of the resulting solid is

$$V = \int_a^b 2\pi x f(x)\, dx$$

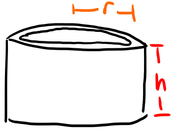
This is the method of shells. It is derived by taking a partition of the x-axis and rotating strips that are parallel to the axis of rotation (i.e. vertical strips).

We can derive a similar formula for the method of shells if the axis of rotation is the x-axis. This requires *horizontal* strips—a partitioning of the y-axis.

The region bounded by $y = \sqrt{x}$, the y-axis and the line $y = 2$ is rotated about the x-axis. Use shells to find the volume of the resulting solid.



$y = \sqrt{x} \Rightarrow x = y^2$



Thickness Δy

$h = y^2$ $\quad r = y$

Volume of one shell

$$V_s = 2\pi r h \Delta y$$

$$= 2\pi y (y^2) \Delta y$$

Volume

$$V = \int_0^2 2\pi\, y \cdot y^2\, dy = 2\pi \int_0^2 y^3\, dy$$

$$= 2\pi \left[\frac{y^4}{4} \right|_0^2$$

$$= 2\pi \left[\frac{2^4}{4} - 0 \right] = 8\pi$$

This is a hyperlink.

Here is a link to an interactive shell method applet.

Overview Solid Formed by Revolution

Method of Disks or Washers: Take cross sections **perpendicular** to the axis of rotation. If the axis of rotation is the x-axis (or y-axis) then

$$\text{Solid of Revolution:} \quad V = \int_a^b \pi (f(x))^2\, dx \quad \left(\text{or} \quad V = \int_c^d \pi (f(y))^2\, dy \right).$$

This can be adjusted as needed for washers (disk with a disk removed) or for other horizontal or vertical axes of rotation.

Solid Formed by Revolution

Method of Cylindrical Shells: Take cross sections **parallel** to the axis of rotation. For shells of thickness Δr, radius r and height h, the volume is

$$\text{Solid of Revolution:} \quad V = \int_a^b 2\pi r h\, dr.$$

For a vertical axis of rotation, $dr = dx$ and for a horizontal one $dr = dy$. Both r and h must be expressed in terms of the variable of integration.

Solid Formed by Revolution

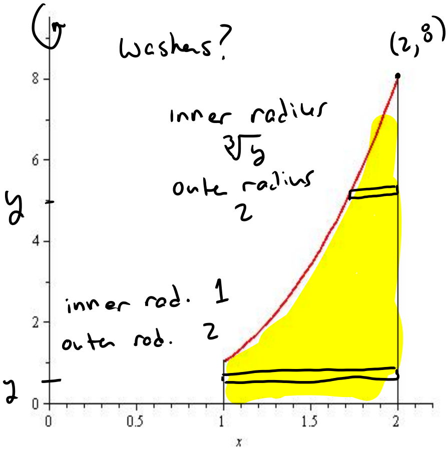
Figure: Performing Arts Center, Kansas City MO

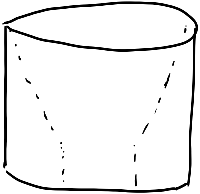
Solid Formed by Revolution

Figure: Performing Arts Center, Kansas City MO

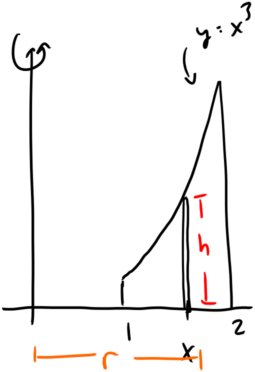
Find the volume of the solid...

The region bound between $y = x^3$, $y = 0$, $x = 1$ and $x = 2$ is rotated about the y-axis.





$$V = \int_0^1 \pi(2^2 - 1^2)\,dy + \int_1^8 \pi\left(2^2 - (\sqrt[3]{y})^2\right)dy$$



For shells

$$h = x^3$$

$$r = x$$

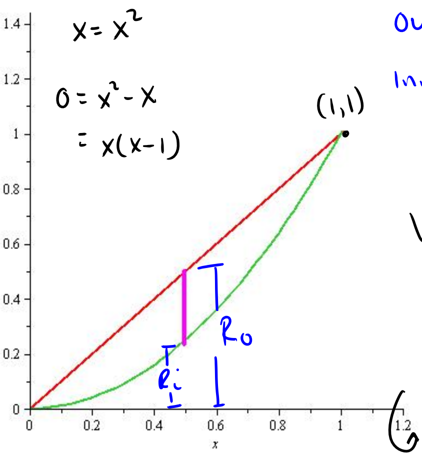
$$V = \int_1^2 2\pi x (x^3)\, dx$$

$$= 2\pi \int_1^2 x^4\, dx = 2\pi \left.\frac{x^5}{5}\right|_1^2$$

$$= 2\pi \left(\frac{2^5}{5} - \frac{1^5}{5}\right) = 2\pi \left(\frac{32-1}{5}\right) = \frac{62\pi}{5}$$

Find the volume of the solid using disks (washers)

The region bound between$y = x$ and $y = x^2$ is rotated about the x-axis.



outer radius $R_o = x$

Inner radius $R_i = x^2$

$$V = \int_0^1 \pi\left(x^2 - (x^2)^2\right) dx$$

Figure: Cross section perpedicular to axis for disks/washers.

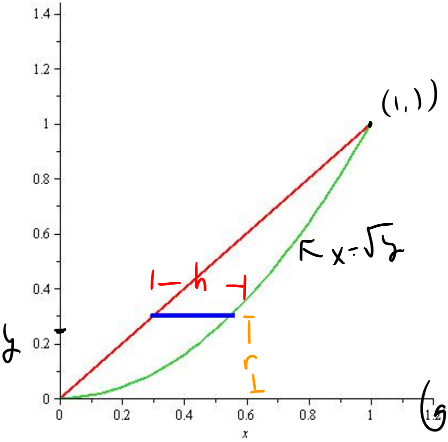
$$V = \pi \int_0^1 (x^2 - x^4)\, dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right|_0^1 = \pi \left[\frac{1^3}{3} - \frac{1^5}{5} - 0 \right]$$

$$= \pi \left(\frac{1}{3} - \frac{1}{5} \right) = \pi \left(\frac{5-3}{15} \right) = \frac{2\pi}{15}$$

Find the volume of the solid using shells.

The region bound between$y = x$ and $y = x^2$ is rotated about the x-axis.



radius $\quad r = y$

height $\quad h = \sqrt{y} - y$

$$V = \int_0^1 2\pi y\left(\sqrt{y} - y\right)\,dy$$

Figure: Cross section parallel to axis for shells.

$$V = \int_0^1 2\pi \left(y^{3/2} - y^2 \right) dy$$

$$= 2\pi \left[\frac{y^{5/2}}{5/2} - \frac{y^3}{3} \right|_0^1$$

$$= 2\pi \left[\frac{2}{5}\left(1^{5/2}\right) - \frac{1^3}{3} - 0 \right] = 2\pi \left(\frac{2}{5} - \frac{1}{3} \right)$$

$$= 2\pi \left[\frac{6-5}{15} \right] = 2\pi\left(\frac{1}{15}\right) = \frac{2\pi}{15}$$