June 7 Math 1190 sec. 51 Summer 2017

Section 1.4: Limits and Continuity of Trigonometric, Exponential and Logarithmic Functions

Theorem: (Squeeze Theorem) Suppose $f(x) \leq g(x) \leq h(x)$ for all x in an interval containing c except possibly at c. If

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$$

then

$$\lim_{x\to c} g(x) = L.$$

Squeeze Theorem:

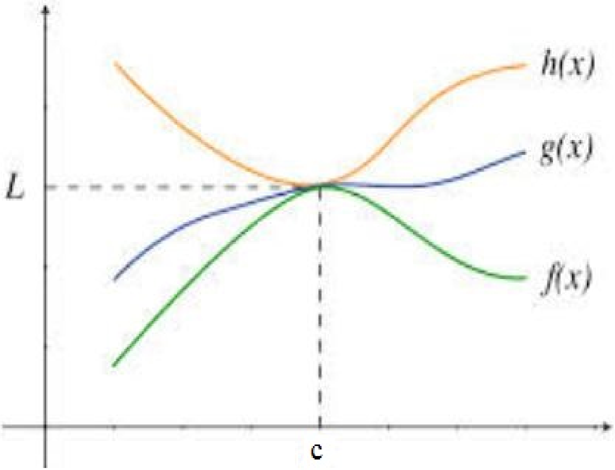


Figure: Graphical Representation of the Squeeze Theorem.

Example: Evaluate

$$\lim_{\theta\to 0}\theta^2\sin\frac{1}{\theta}$$

Since "$\frac{1}{0}$" isn't defined, we can't just plug in 0.

i.e. $\sin\frac{1}{\theta}$ doesn't make sense if $\theta=0$.

We'll use the squeeze theorem and the property of the sine

$$-1\le \sin\frac{1}{\theta}\le 1$$

We want $\theta^2\sin\frac{1}{\theta}$ to be the "g" in our inequality. Using $\theta^2\ge 0$, multiply through to get $$-1\cdot\theta^2 \le \theta^2\sin\frac{1}{\theta} \le 1\cdot\theta^2$$

$$\Rightarrow \quad -\theta^2 \le \theta^2\sin\frac{1}{\theta} \le \theta^2$$

form $f(\theta) \le g(\theta) \le h(\theta)$

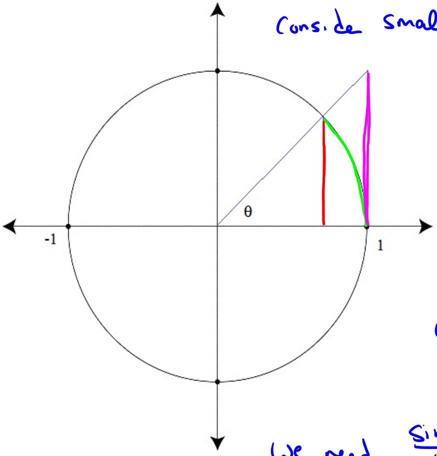
Note $\lim_{\theta\to 0} -\theta^2 = -0^2 = 0$ and $\lim_{\theta\to 0} \theta^2 = 0^2 = 0$

By the squeeze theorem, it must be that

$$\lim_{\theta\to 0} \theta^2\sin\frac{1}{\theta} = 0$$ too.

A Couple of Important Limits

Theorem: $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ and $\displaystyle\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0$



Consider small, positive θ

Consider three lengths

- $\sin\theta$
- θ (arc length $= r\theta$)
- $\tan\theta$

We see that

$$\sin\theta \le \theta \le \tan\theta$$

We need $\dfrac{\sin\theta}{\theta}$ trapped between 2 other things.

$\sin\theta \le \theta$ for θ small, positive divide by θ

$$\frac{\sin\theta}{\theta} \le \frac{\theta}{\theta} = 1 \quad \Rightarrow \quad \frac{\sin\theta}{\theta} \le 1$$

Recall, sine is odd. i.e. $\sin(-\theta) = -\sin\theta$

Replacing θ with $-\theta$ we get

$$\frac{\sin(-\theta)}{-\theta} = \frac{-\sin\theta}{-\theta} = \frac{\sin\theta}{\theta}$$

So $\frac{\sin\theta}{\theta} \le 1$ for θ small positive or small negative.

We also had $\theta \le \tan\theta$ for θ small positive.

$\theta \le \dfrac{\sin\theta}{\cos\theta}$ divide by θ, multiply by $\cos\theta$

$\cos\theta \le \dfrac{\sin\theta}{\theta}$

We already know that $\dfrac{\sin(-\theta)}{-\theta} = \dfrac{\sin\theta}{\theta}$.

Recall that $\cos\theta$ is even.

i.e. $\cos(-\theta) = \cos\theta$

So $\cos\theta \le \dfrac{\sin\theta}{\theta}$ for θ small positive and θ small negative.

Combining these we have

$$\cos\theta \le \frac{\sin\theta}{\theta} \le 1 \quad \text{for } \theta \text{ near zero.}$$

$$\lim_{\theta\to 0} \cos\theta = \cos 0 = 1 \quad \text{and} \quad \lim_{\theta\to 0} 1 = 1$$

By the squeeze theorem

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 .$$

Observation

Our result $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ can be stated with any variable name. What is critical is

- The argument of the sine must match exactly the denominator, and
- this argument must be tending to zero.

Hence, the following are all true

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \lim_{y\to 0} \frac{\sin y}{y} = 1, \quad \lim_{\heartsuit\to 0} \frac{\sin 3\heartsuit}{3\heartsuit} = 1$$

$$\lim_{x\to 7} \frac{\sin x}{x} = \frac{\sin 7}{7}$$

Observation

Since 1 is its own reciprocal, it is also true that

$$\lim_{\theta \to 0} \frac{\theta}{\sin \theta} = 1.$$

Similarly,

$$\lim_{x \to 0} \frac{x}{\sin x} = 1, \quad \lim_{y \to 0} \frac{y}{\sin y} = 1, \quad \lim_{\heartsuit \to 0} \frac{3\heartsuit}{\sin 3\heartsuit} = 1$$

If the limit doesn't match the form, care must be taken. For example, none of the following limits is 1.

$$\lim_{x \to 0} \frac{\sin(4x)}{x}, \quad \lim_{t \to 0} \frac{\cos t}{t}, \quad \lim_{\heartsuit \to 0} \frac{3\heartsuit}{\sin^2 3\heartsuit}$$

Examples

Evaluate each limit if possible.

(a) $\displaystyle\lim_{x\to 0} \frac{\sin(4x)}{x}$

we'll use $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

It is true that $\displaystyle\lim_{x\to 0} \frac{\sin(4x)}{4x} = 1$

we'll get $4x$ in the denominator by multiplying by $1 = \frac{4}{4}$

$$\lim_{x\to 0} \frac{\sin(4x)}{x} = \lim_{x\to 0} \frac{\sin(4x)}{x} \cdot \frac{4}{4}$$

$$= \lim_{x\to 0} 4\, \frac{\sin(4x)}{4x} = 4 \lim_{x\to 0} \frac{\sin(4x)}{4x}$$

$$= 4 \cdot 1 = 4$$

(b) $\lim_{t\to 0} \frac{\csc(3t)}{\csc t}$ $= \lim_{t\to 0} \frac{\frac{1}{\sin(3t)}}{\frac{1}{\sin t}} = \lim_{t\to 0} \frac{\sin t}{\sin(3t)}$

$$= \lim_{t\to 0} \frac{\sin t}{1} \frac{1}{\sin(3t)} \left(\frac{3t}{3t}\right)$$

$$= \lim_{t\to 0} \frac{1}{3} \left(\frac{\sin t}{t}\right)\left(\frac{3t}{\sin(3t)}\right)$$

$$= \frac{1}{3} \cdot (1) \cdot (1) = \frac{1}{3}$$

Questions

(1) Evaluate if possible $\lim_{x \to 0} \frac{\tan(2x)}{4x}$ $= \lim_{x \to 0} \frac{\frac{\sin(2x)}{\cos(2x)}}{4x}$

Hint: $\tan(2x) = \frac{\sin(2x)}{\cos(2x)}$ and $\cos(0) = 1$

(a) $\frac{1}{4}$

(b) $\frac{1}{2}$

(c) 2

(d) DNE

$= \lim_{x \to 0} \frac{\sin(2x)}{4x} \cdot \frac{1}{\cos(2x)}$

$= \lim_{x \to 0} \frac{1}{2} \frac{\sin(2x)}{2x} \cdot \frac{1}{\cos(2x)}$

$= \frac{1}{2} \cdot 1 \cdot \frac{1}{1} = \frac{1}{2}$

Question

Evaluate the limit

$$\lim_{\theta \to 0} \frac{\cos(7\theta)}{\cos(2\theta)} \quad = \quad \frac{\cos(0)}{\cos(0)} = 1$$

(a) $\frac{7}{2}$

(b) 1

(c) 0

(d) DNE

Section 1.5: Infinite Limits, Limits at Infinity, Asymptotes

Here, we will consider limits involving the symbols ∞ (*infinity*) and $-\infty$ (*negative infinity*). They will be used to denote **unboundedness** in the positive and negative directions, respectively.

While ∞ and $-\infty$ are NOT NUMBERS, there is an arithmetic for using these symbols. In particular

- $\infty + \infty = \infty$
- $\infty + c = \infty$ for any real number c
- $\infty \cdot c = \infty$ if $c > 0$ and $\infty \cdot c = -\infty$ if $c < 0$
- $\frac{0}{\infty} = \frac{0}{-\infty} = 0$

Other forms that may appear are indeterminate. The following **are not defined.**

$$\infty - \infty, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty^0$$

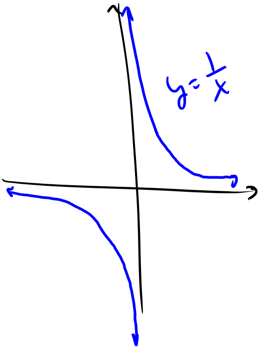
Infinite Limits

Investigate the limit

$$\lim_{x \to 0^-} \frac{1}{x}$$

x	$f(x) = \frac{1}{x}$
-0.1	-10
-0.01	-100
-0.001	-1000
0	undefined

look unboundedly large and negative



we can write this as $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

Infinite Limits

Investigate the limit

$$\lim_{x\to 0^+} \frac{1}{x}$$

x	$f(x) = \frac{1}{x}$
0.1	10
0.01	100
0.001	1000
0	undefined

get unboundedly large

We can write this as

$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

Note: since $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

It is the case that

$\lim_{x\to 0} \frac{1}{x}$ DNE

Infinite Limits

Definition: Let $f(x)$ be defined on an open interval containing c except possibly at c. Then

$$\lim_{x \to c} f(x) = \infty$$

provided $f(x)$ can be made arbitrarily large by taking x sufficiently close to c. (The definition of

$$\lim_{x \to c} f(x) = -\infty$$

is similar except that f can be made arbitrarily large and negative.)

The top limit statement reads *the limit as x approaches c of $f(x)$ equals infinity*.

An Observation

Suppose when taking a limit $\lim_{x\to c} f(x)$ we see the form

$$\frac{k}{0} \quad \text{where } k \text{ is any nonzero real number.}$$

Then this limit **MAY** be either ∞ or $-\infty$.

- If we determine that the ratio is positive for all x near c, the limit is ∞.
- If we determine that the ratio is negative for all x near c, the limit is $-\infty$.
- If the ratio can take either sign for x sufficiently close to c, the limit DNE.

Evaluate Each Limit if Possible

(a) $\displaystyle\lim_{x\to 1^-} \frac{2x+1}{x-1}$

The form seen here is " $\frac{3}{0}$ ".

We'll do a sign analysis. The numerator goes to 3, and 3 is positive.

Recall, $x \to 1^-$ means x goes to 1 but $x < 1$.

If $x < 1$, then $x - 1 < 0$. So $x-1$ goes to zero through negative numbers.

$\frac{\text{positive}}{\text{negative}}$ = negative

So $\displaystyle\lim_{x\to 1^-} \frac{2x+1}{x-1} = -\infty$

(b) $\lim_{x \to 1^+} \frac{2x+1}{x-1}$

Still looks like "$\frac{3}{0}$".

$x \to 1^+$ means x goes to 1 but $x > 1$.

If $x > 1$ then $x - 1 > 0$.

So $x-1$ goes to zero through positive numbers.

$$\lim_{x \to 1^+} \frac{2x+1}{x-1} = \infty$$

(c) $\lim_{x \to 1} \frac{2x+1}{x-1}$ DNE

since it goes to $+\infty$ on the

right and $-\infty$ on the

left.

(d) $\lim_{x\to 3} \frac{x-x^2}{|x-3|}$

The form here is "$\frac{-6}{0}$"

We do the sign analysis. The top, -6, is negative.

Because of the absolute value bars, $|x-3|$ is positive for all x close to 3. So $|x-3|$ goes to zero through positive numbers.

$$\lim_{x\to 3} \frac{x-x^2}{|x-3|} = -\infty$$

Question

Evaluate if possible $\lim_{t \to 2} \frac{4}{(t-2)^2}$

4 - positive

$(t-2)^2 \to 0$

but $(t-2)^2$ is positive.

(a) ∞

(b) $-\infty$

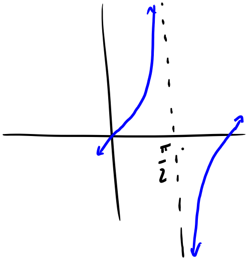
(c) 4

(d) DNE

Well Known Infinite Limits

Some limits that follow from what we know about these functions

$$\lim_{x\to 0^+} \ln(x) = -\infty$$



$$\lim_{\theta\to \frac{\pi}{2}^-} \tan\theta = \infty \quad \text{and} \quad \lim_{\theta\to \frac{\pi}{2}^+} \tan\theta = -\infty$$

$$\lim_{\theta\to 0^-} \cot\theta = -\infty \quad \text{and} \quad \lim_{\theta\to 0^+} \cot\theta = \infty$$

$$\lim_{\theta\to \frac{\pi}{2}^-} \sec\theta = \infty \quad \text{and} \quad \lim_{\theta\to \frac{\pi}{2}^+} \sec\theta = -\infty$$

$$\lim_{\theta\to 0^-} \csc\theta = -\infty \quad \text{and} \quad \lim_{\theta\to 0^+} \csc\theta = \infty$$

Vertical Asymptotes

Definition The line $x = c$ is a *vertical asymptote* to the graph of f if

$$\lim_{x\to c^+} f(x) = \pm\infty, \quad \text{or} \quad \lim_{x\to c^-} f(x) = \pm\infty.$$

A good candidate for the location of a vertical asymptote is a value that makes a denominator zero.

Limits at Infinity

We know what is meant by a limit being infinite (i.e. $f \to \infty$ or $f \to -\infty$). Now, we want to consider limits like

$$\lim_{x\to\infty} f(x) \qquad \text{or like}$$

$$\lim_{x\to-\infty} f(x).$$

What is meant by such a thing, and how is it related to a function's graph?

Definitions

Let f be defined on an interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

provided the value of f can be made arbitrarily close to L by taking x sufficiently large.

Similarly

Defintion: Let f be defined on an interval $(-\infty, a)$. Then

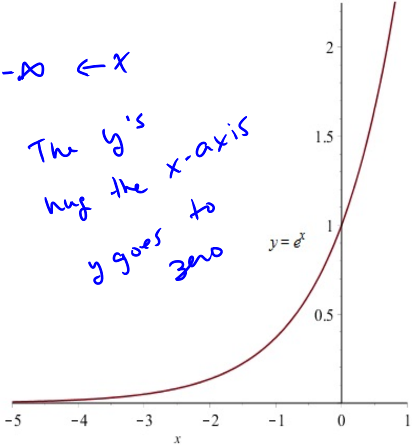
$$\lim_{x\to-\infty} f(x) = L$$

provided the value of f can be made arbitrarily close to L by taking x sufficiently large and negative.

Example

Investigate the limit

$$\lim_{x \to -\infty} e^x$$



$-\infty \leftarrow x$

The y's hug the x-axis

y goes to zero

$$\lim_{x \to -\infty} e^x = 0$$

Horizontal Asymptotes

Definition: The line $y = L$ is a *horizontal asymptote* to the graph of f if

$$\lim_{x\to\infty} f(x) = L, \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

Some Results to Remember

Let k be any real number and let p be positive and rational. Then

$$\lim_{x\to\infty} \frac{k}{x^p} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{k}{x^p} = 0$$

The latter holds assuming x^p is defined for $x < 0$.

$$\lim_{x\to\infty} \frac{3}{x^7} = 0, \qquad \lim_{x\to-\infty} \frac{-\pi}{\sqrt[3]{x}} = 0$$

$$\lim_{x\to\infty} \frac{17}{x^{3/5}} = 0$$

In essence, if the numerator of a ratio is staying finite while the denominator is becoming infinite, the ratio is tending to zero.

Examples

Evaluate if possible

$$\lim_{x\to\infty} \frac{3x^2+2x-1}{x^2+5x+2}$$

- Identify the largest power in the denominator p

- Multiply by $1 = \frac{\frac{1}{x^p}}{\frac{1}{x^p}}$

$$= \lim_{x\to\infty} \left(\frac{3x^2+2x-1}{x^2+5x+2}\right)\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{3x^2}{x^2}+\frac{2x}{x^2}-\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{5x}{x^2}+\frac{2}{x^2}}$$

$$= \lim_{x\to\infty} \frac{3+\frac{2}{x}-\frac{1}{x^2}}{1+\frac{5}{x}+\frac{2}{x^2}}$$

$$= \frac{3 + 0 - 0}{1 + 0 + 0} = 3$$

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1}$$

We can play the same game.

Here $p=1$.

We'll use that $\sqrt{x^2} = x$ if $x>0$.

So $\frac{1}{x} = \frac{1}{\sqrt{x^2}}$

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1} = \lim_{x\to\infty} \left(\frac{\sqrt{x^2+1}}{x+1}\right) \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}\sqrt{x^2+1}}{1+\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x^2}}\sqrt{x^2+1}}{1+\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^2}(x^2+1)}}{1+\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x^2}}}{1+\frac{1}{x}} = \frac{\sqrt{1+0}}{1+0} = 1$$

* Note, if we were taking $x \to -\infty$

we'd use $\sqrt{x^2} = -x$ so $\frac{1}{x} = \frac{-1}{\sqrt{x^2}}$

$|x| = -x$

Question

Evaluate if possible

$$\lim_{x\to\infty} \frac{\sqrt{3x^2+2x}}{4x+3}$$

mult. by $\frac{\frac{1}{x}}{\frac{1}{x}}$

use $\frac{1}{x} = \frac{1}{\sqrt{x^2}}$

(a) DNE

(b) $\frac{3}{4}$

(c) $\sqrt{3}$

(d) $\frac{\sqrt{3}}{4}$

Infinte Limits at Infinity

The following limits may arise

$$\lim_{x\to\infty} f(x) = \infty, \qquad \lim_{x\to\infty} f(x) = -\infty$$

$$\lim_{x\to-\infty} f(x) = \infty, \qquad \lim_{x\to-\infty} f(x) = -\infty$$

Two critical limits to remember (YOU'LL NEED TO KNOW THESE)

$$\lim_{x\to\infty} e^x = \infty \quad \text{and} \quad \lim_{x\to\infty} \ln(x) = \infty$$

Questions

(1) **True or False:** Since $\lim_{x \to 0^+} \ln(x) = -\infty$, we can conclude that the line $x = 0$ is a vertical asymptote to the graph of $y = \ln(x)$.

(2) **True or False:** Since $\lim_{x \to -\infty} e^x = 0$, we can conclude that the line $y = 0$ is a horizontal asymptote to the graph of $y = e^x$.

Section 2.1: Rates of Change and the Derivative

We opened by saying that Calculus is concerned with the way in which quantities change. An obvious example of change is motion of an object in space (change of position).

Here we introduce the idea of *rate of change* and the mathematical formulation of this called a *derivative*.

Though we'll use **rectilinear motion** (i.e. movement along a straight line) as an illustrative example, the concept can be applied to many processes in physics, chemistry, biology, business, and the list goes on!

Motivational Example:

Suppose a ball is dropped from the top of the Space Needle 605 feet high. According to Galileo's law, the distance $s(t)$ feet the ball has fallen after t seconds is (neglecting wind drag)

$$s = f(t) = 16t^2.$$

The position of the ball relative to the top of the tower is changing. We can consider the ball's velocity.

We define **average velocity** as

$$\text{change in position} \quad \div \quad \text{change in time.}$$

average velocity = change in position $\div$ change in time

Find the average velocity over the period from $t = 0$ to $t = 2$.

$s = f(t) = 16t^2$

So @ $t = 0$ $\quad s = f(0) = 16 \cdot 0^2 = 0$ ft

$t = 2$ $\quad s = f(2) = 16 \cdot 2^2 = 64$ ft

Average velocity is $\dfrac{64\,\text{ft} - 0\,\text{ft}}{2\,\text{sec} - 0\,\text{sec}} = \dfrac{64}{2} \dfrac{\text{ft}}{\text{sec}}$

$= 32 \dfrac{\text{ft}}{\text{sec}}.$

average velocity = change in position $\div$ change in time

Find the average velocity over the period from $t = 2$ to $t = 4$.

@ $t=2$, $s = 64$ ft

$t=4$, $s = f(4) = 16(4^2) = 256$ ft

over this interval

$$\text{avg. velocity} = \frac{256\,\text{ft} - 64\,\text{ft}}{4\,\text{sec} - 2\,\text{sec}} = \frac{192\,\text{ft}}{2\,\text{sec}}$$

$$= 96\,\frac{\text{ft}}{\text{sec}}$$

Here's a tougher question...

What is the *instantaneous velocity* when $t = 2$?

we only have one time / position pair.

Let's choose another time $t \neq 2$ (maybe $t \approx 2$)

The average velocity from time t to time 2

is

$$\frac{f(t) - f(2)}{t-2} = \frac{f(t) - 64 \text{ ft}}{t-2} \frac{\text{ft}}{\text{sec}}$$

$$= \frac{16t^2 - 64}{t-2} \ \frac{\text{ft}}{\text{sec}}$$

Estimating instantaneous velocity using intervals of decreasing size...

here $\Delta t = t - 2$ change in t

$\Delta s = f(t) - f(2)$ change in S.

Δt	$\frac{\Delta s}{\Delta t} = \frac{f(t)-f(2)}{t-2}$	Δt	$\frac{\Delta s}{\Delta t} = \frac{f(t)-f(2)}{t-2}$
1	80	-1	48
0.1	65.6	-0.1	62.4
0.05	64.8	-0.05	63.2
0.01	64.16	-0.01	63.84

Looks like the velocity is $64 \frac{ft}{sec}$.

look to be approaching 64

Instantaneous Velocity

If we consider the independent variable t and dependent variable $s = f(t)$, we note that the velocity has the form

$$\frac{\text{change in } s}{\text{change in } t} = \frac{\Delta s}{\Delta t}$$

Definition: We define the instantaneous velocity v (simply called *velocity*) at the time t_0 as

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$$

provided this limit exists.

Example

An object moves along the x-axis such that its distance s from the origin at time t is given by $s = \sqrt{2t}$. If s is in inches and t is in seconds, determine the object's velocity at $t = 3$ sec.

Here $f(t) = \sqrt{2t}$ and $t_0 = 3$.

$$f(t_0) = f(3) = \sqrt{2 \cdot 3} = \sqrt{6}$$

velocity v

$$v = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$$

$$= \lim_{t \to 3} \frac{\sqrt{2t} - \sqrt{6}}{t - 3}$$

well use the conjugate

$$= \lim_{t\to 3} \left(\frac{\sqrt{2t}-\sqrt{6}}{t-3}\right)\left(\frac{\sqrt{2t}+\sqrt{6}}{\sqrt{2t}+\sqrt{6}}\right)$$

$$= \lim_{t\to 3} \frac{2t-6}{(t-3)(\sqrt{2t}+\sqrt{6})}$$

$$= \lim_{t\to 3} \frac{2\cancel{(t-3)}}{\cancel{(t-3)}(\sqrt{2t}+\sqrt{6})}$$

$$= \lim_{t\to 3} \frac{2}{\sqrt{2t}+\sqrt{6}} = \frac{2}{\sqrt{2\cdot 3}+\sqrt{6}}$$

$$= \frac{2}{\sqrt{6}+\sqrt{6}} = \frac{2}{2\sqrt{6}} = \frac{1}{\sqrt{6}}$$

The velocity at 3 seconds is $\frac{1}{\sqrt{6}}\ \frac{\text{in}}{\text{sec}}$.

Question

A cannon ball is fired from the ground so that it's distance from the ground after t seconds is given by $s = 80t - 16t^2$ feet. Which of the following limits would be used to determine the ball's velocity at $t = 3$ seconds?

(a) $\displaystyle\lim_{t\to 0} \frac{80t - 16t^2 - 96}{t}$

(b) $\displaystyle\lim_{t\to 3} \frac{80t - 16t^2 - 96}{t-3}$

(c) $\displaystyle\lim_{t\to 0} \frac{80t - 16t^2 - 96}{t-3}$

(d) $\displaystyle\lim_{t\to 3} \frac{80t - 16t^2 - 96}{t}$

Observation

Note that the average velocity has the form $\frac{\Delta s}{\Delta t}$. This ratio (should) look familiar. If we think graphically, with $s = f(t)$

$$\frac{\Delta s}{\Delta t} = \frac{\text{rise}}{\text{run}} = \text{slope}$$

The Tangent Line Problem

Given a graph of a function $y = f(x)$:

A **secant** line is a line connecting two points $P = (x_0, y_0)$ and $Q = (x_1, y_1)$ on the graph. The slope of a secant line is

$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Recall that if $P = (c, f(c))$ and $Q = (x, f(x))$ are distinct points, we denoted the slope of the secant line

$$m_{sec} = \frac{f(x) - f(c)}{x - c}$$

We had defined the slope of the tangent line as

$$m_{tan} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{if this limit exists.}$$

Example

Find the slope of the line tangent to the graph of $y = \frac{1}{x}$ at the point $(2, \frac{1}{2})$.

$$m_{tan} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}, \quad f(x) = \frac{1}{x} \text{ and } c = 2$$

$$m_{tan} = \lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2}$$

$$= \lim_{x \to 2} \frac{\frac{2}{2x} - \frac{x}{2x}}{x - 2}$$

$$= \lim_{x \to 2} \frac{\frac{2 - x}{2x}}{x - 2}$$

$$= \lim_{x \to 2} \frac{2-x}{2x} \cdot \frac{1}{x-2}$$

$$= \lim_{x \to 2} \frac{2-x}{2x(x-2)}$$

$$= \lim_{x \to 2} \frac{-\cancel{(x-2)}}{2x\cancel{(x-2)}} = \lim_{x \to 2} \frac{-1}{2x} = \frac{-1}{2(2)} = \frac{-1}{4}$$

The slope $m_{tan} = \frac{-1}{4}$ @ $(2, \frac{1}{2})$.

Example Continued...

Find the equation of the line tangent to the graph of $y = \frac{1}{x}$ at the point $(2, \frac{1}{2})$.

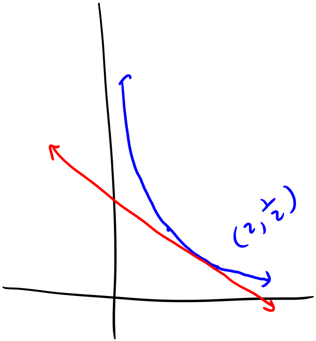
The slope $m = \frac{-1}{4}$. Using point/slope form

$y - y_0 = m(x - x_0)$

$$y - \frac{1}{2} = \frac{-1}{4}(x-2)$$

$$y = \frac{-1}{4}x + \frac{1}{2} + \frac{1}{2}$$

$$y = \frac{-1}{4}x + 1$$



Tangent Line

Theorem: Let $y = f(x)$ and let c be in the domain of f. If the slope m_{tan} exists at the point $(c, f(c))$, then the equation of the line tangent to the graph of f at this point is

$$y = m_{tan}(x - c) + f(c).$$

The Derivative

Let $y = f(x)$. For $x \neq c$ we'll call $\frac{f(x)-f(c)}{x-c}$ the average rate of change of f on the interval from x to c.

We'll call

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} \quad \text{the rate of change of } f \text{ at } c$$

if this limit exists.

Definition: Let $y = f(x)$ at let c be in the domain of f. The **derivative** of f at c is denoted $f'(c)$ and is defined as

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

provided the limit exists.

The Derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

In addition to *the derivative of f at c*, the notation $f'(c)$ is read as

- f prime of c, or
- f prime at c.

At this point, we have several interpretations of this same **number** $f'(c)$.

- as a velocity if f is the position of a moving object,
- as a rate of change of the function f when $x = c$,
- as the slope of the line tangent to the graph of f at $(c, f(c))$.

Question

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Determine $f'(c)$ if $f(x) = 2x - x^2$ and $c = 1$.

(a) $f'(1) = 2 - 2x$

$f(1) = 2 \cdot 1 - 1^2 = 1$

(b) $f'(1) = 2$

$f'(1) = \lim_{x \to 1} \frac{2x - x^2 - 1}{x - 1}$

(c) $f'(1)$ DNE

$\vdots$

(d) $f'(1) = 0$

$= 0$

Section 2.2: The Derivative as a Function

If $f(x)$ is a function, then the set of numbers $f'(c)$ for various values of c can define a new function. To proceed, we consider an alternative formulation for $f'(c)$.

If it exists, then $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$. Let $h = x - c$. Then $h \to 0$ if $x \to c$, and $x = c + h$. Hence we can write $f'(c)$ as

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

If $h = x - c$, then $x = c + h$

The Derivative Function

Let f be a function. Define the new function f' by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

called the **derivative** of f. The domain of this new function is the set

$$\{x | x \text{ is in the domain of } f, \text{ and } f'(x) \text{ exists}\}.$$

f' is read as "f prime."