June 8 Math 2254 sec 001 Summer 2015

Section 6.5: Arclength

In this section, we consider a segment of a continuous curve $y = f(x)$ over an interval $[a, b]$. If we think of the curve as a very thin wire, we can ask what its length is.

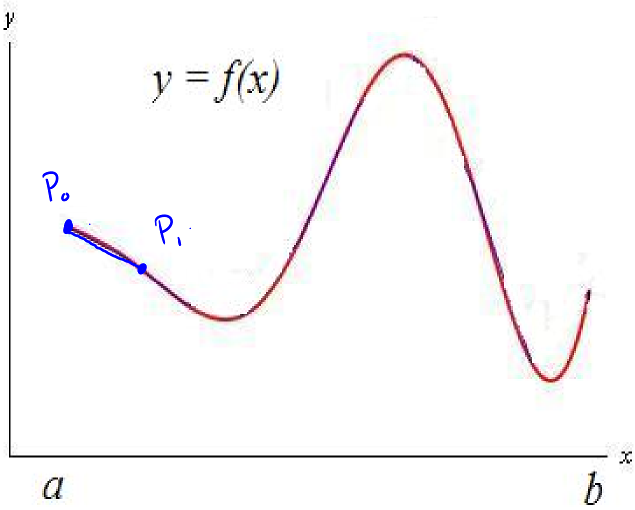


Figure: A continuous curve $y = f(x)$ over the interval $a \leq x \leq b$ has some length we wish to determine.

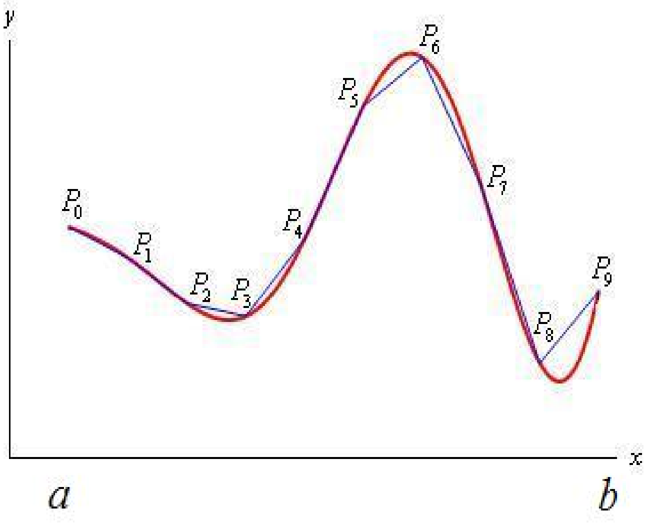
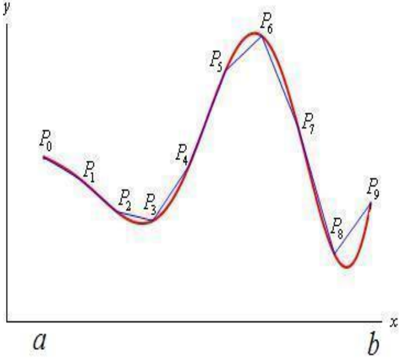


Figure: We can approximate its length by picking some points on the curve and connecting them with straight lines. Then the sum of the lengths of these line segments approximates the length of the actual curve.



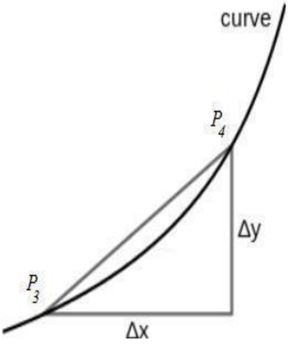
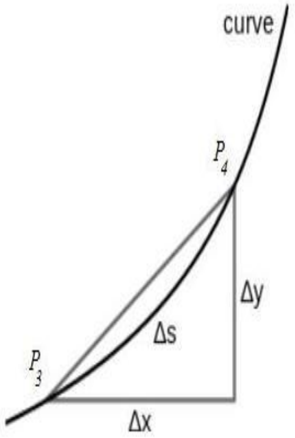


Figure: To find the length of one such line segment, we can use the Pythagorean Theorem. Here, we pick the segment between points P_3 and P_4 and take a closer look.



$$|\overline{P_3P_4}|^2 = (\Delta x)^2 + (\Delta y)^2$$

$$(\Delta s)^2 \approx (\Delta x)^2 + (\Delta y)^2$$

$$\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Figure: If Δs is the *true* arclength, we can see what its approximation is. $\Delta s \approx |\overline{P_3 P_4}|$.

$$\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(\Delta x)^2\left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)}$$

$$= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

We'll take the limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

Arclength formula

Suppose the function $y = f(x)$ has continuous derivative f' on the interval $[a, b]$. The arclength s of the continuous curve $y = f(x)$ on the interval $[a, b]$ is

$$s = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

$\frac{dy}{dx} = f'(x)$

It is worth noting that this formula often leads to integrals that can not be evaluated by hand (*often*, but not *always*).

Example

Find the arclength of the curve $y = \ln(\csc x)$ from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{2}$.

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{-\csc x \cot x}{\csc x} = -\cot x$$

$*$ $1 + \cot^2 x = \csc^2 x$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + (-\cot x)^2 = 1 + \cot^2 x$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \cot^2 x} = \sqrt{\csc^2 x} = |\csc x|$$

$$= \csc x \quad \text{for} \quad \pi/4 \le x \le \frac{\pi}{2}$$

$$S = \int_{\pi/4}^{\pi/2} \csc x \, dx = -\ln|\csc x + \cot x| \Big|_{\pi/4}^{\pi/2}$$

$$= -\ln\left|\csc \pi/2 + \cot \frac{\pi}{2}\right| - \left(-\ln\left|\csc \frac{\pi}{4} + \cot \frac{\pi}{4}\right|\right)$$

$$= -\ln|1| + \ln|\sqrt{2} + 1|$$

$$= \ln|\sqrt{2} + 1|$$

Example

Find the length of the curve $y = \frac{3}{2}x^{2/3}$ on the interval from $x = 1$ to $x = 8$.

$$S = \int_a^b \sqrt{1 + (y')^2}\, dx$$

$$y' = \frac{3}{2}\left(\frac{2}{3}x^{-1/3}\right) = x^{-1/3} \Rightarrow 1 + (y')^2 = 1 + \left(x^{-1/3}\right)^2$$

$$1 + (y')^2 = 1 + x^{-2/3} = 1 + \frac{1}{x^{2/3}} = \frac{x^{2/3}+1}{x^{2/3}}$$

$$\sqrt{1 + (y')^2} = \sqrt{\frac{x^{2/3}+1}{x^{2/3}}} = \frac{\sqrt{x^{2/3}+1}}{\sqrt{x^{2/3}}} = \frac{1}{x^{1/3}}\sqrt{x^{2/3}+1}$$

$$S = \int_1^8 \frac{1}{x^{1/3}} \sqrt{x^{2/3}+1}\, dx$$

Let $u = x^{2/3}+1$

$du = \frac{2}{3} x^{-1/3}\, dx$

$\frac{3}{2}\, du = x^{-1/3}\, dx$

If $x=1$, $u = 1^{2/3}+1 = 2$

$x=8$, $u = 8^{2/3}+1 = 5$

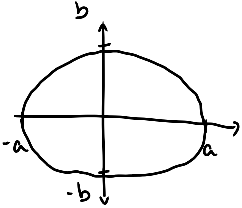
$$= \int_2^5 \frac{3}{2}\sqrt{u}\, du$$

$$= \frac{3}{2}\int_2^5 u^{1/2}\, du$$

$$= \frac{3}{2}\, \frac{u^{3/2}}{3/2}\Big|_2^5 = 5^{3/2} - 2^{3/2} = 5\sqrt{5} - 2\sqrt{2}$$

Example

Find an integral[1] for the perimeter of an ellipse



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solve for top half

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow \frac{y}{b} = \sqrt{1 - \frac{x^2}{a^2}}$$

$$\Rightarrow y = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$S = \int_A^B \sqrt{1 + (y')^2}\, dx$$

[1] We won't try to evaluate it.

$$\frac{dy}{dx} = b\,\frac{1}{2}\left(1-\frac{x^2}{a^2}\right)^{-1/2}\left(-\frac{2x}{a^2}\right) = -\frac{b}{a^2}\,x\,\frac{1}{\sqrt{1-x^2/a^2}}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{-bx}{a^2}\,\frac{1}{\sqrt{1-x^2/a^2}}\right)^2 = 1+\frac{b^2x^2}{a^4\left(1-\frac{x^2}{a^2}\right)}$$

$$= 1+\frac{b^2x^2}{a^4-a^2x^2} = \frac{a^4-a^2x^2+b^2x^2}{a^4-a^2x^2}$$

Our perimeter is

$$\boxed{s = 2\int_{-a}^{a}\sqrt{\frac{a^4-a^2x^2+b^2x^2}{a^4-a^2x^2}}\,dx}$$

Example

Find the length of the curve $y = \frac{x^3}{6} + \frac{1}{2x}$ from $x = 1$ to $x = 3$

$$s = \int_a^b \sqrt{1 + (y')^2}\, dx$$

$$y' = \frac{3x^2}{6} + \frac{1}{2}\left(-x^{-2}\right) = \frac{x^2}{2} - \frac{1}{2x^2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$$

$$1 + (y')^2 = 1 + \left[\frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)\right]^2 = 1 + \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right)$$

$$= \frac{1}{4}\left(4 + x^4 - 2 + \frac{1}{x^4}\right) = \frac{1}{4}\left(x^4 + 2 + \frac{1}{x^4}\right)$$

$$= \frac{1}{4}\left(x^2 + \frac{1}{x^2}\right)^2$$

So $\sqrt{1+(y')^2} = \sqrt{\frac{1}{4}\left(x^2+\frac{1}{x^2}\right)^2} = \frac{1}{2}\left|x^2+\frac{1}{x^2}\right| = \frac{1}{2}\left(x^2+\frac{1}{x^2}\right)$

$$s = \int_1^3 \frac{1}{2}\left(x^2+\frac{1}{x^2}\right)dx = \frac{1}{2}\left[\frac{x^3}{3}+\frac{x^{-1}}{-1}\right|_1^3$$

$$= \frac{1}{2}\left[\frac{x^3}{3}-\frac{1}{x}\right|_1^3 = \frac{1}{2}\left[\frac{3^3}{3}-\frac{1}{3}-\left(\frac{1^3}{3}-\frac{1}{1}\right)\right]$$

$$= \frac{1}{2}\left(9-\frac{1}{3}-\frac{1}{3}+1\right) = \frac{1}{2}\left(10-\frac{2}{3}\right) = \frac{1}{2}\ \frac{30-2}{3} = \frac{28}{2\cdot 3} = \frac{14}{3}$$

Section 6.6 Work

The physical concept of *work* is the product of the force applied to an object (maginute in the direction of motion) and the distance through which the object is moved.

$$W = Fd$$

For example, if 15 lb of horizontal force is applied to a wagon that is pulled 30 ft along the horizontal, then the work done is

$$W = 15 \times 30\, ft \cdot lb = 450 ft \cdot lb$$

Work with Variable Force

Suppose we wish to move an object along the x-axis from $x = a$ to $x = b$ applying a variable force $F(x)$. We may begin by forming a partition $\{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ and approximating the total work done by assuming that F is constant on each subinterval. We obtain

$$W \approx F(u_1^*)\Delta x + F(u_2^*)\Delta x + \cdots + F(u_n^*)\Delta x$$

We define the work in the limit as $n \to \infty$ to obtain the integral formula

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} F(u_i^*)\Delta x = \int_a^b F(x)\,dx$$

Units

American Standard: Force is given in pounds, and length in feet. The unit for work is the foot-pound (ft lb).

SI (International): Force is given in Newtons (N), and length is in meters. The unit for work is the Newton-meter which is also called a Joule (J).

Example

A particle is moved along the x-axis from the point $x = 2$ to $x = 5$ by applying a force of $F(x) = x^2 - x$. Find the work done.

$$W = \int_a^b F(x)\,dx \qquad W = \int_2^5 (x^2 - x)\,dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2}\Big|_2^5 = \frac{5^3}{3} - \frac{5^2}{2} - \left(\frac{2^3}{3} - \frac{2^2}{2}\right)$$

$$= \frac{125}{3} - \frac{25}{2} - \frac{8}{3} + \frac{4}{2} = \frac{117}{3} - \frac{21}{2}$$

$$= 39 - \frac{21}{2} = \frac{78-21}{2} = \frac{57}{2}$$

Hooke's Law

Hooke's Law says that

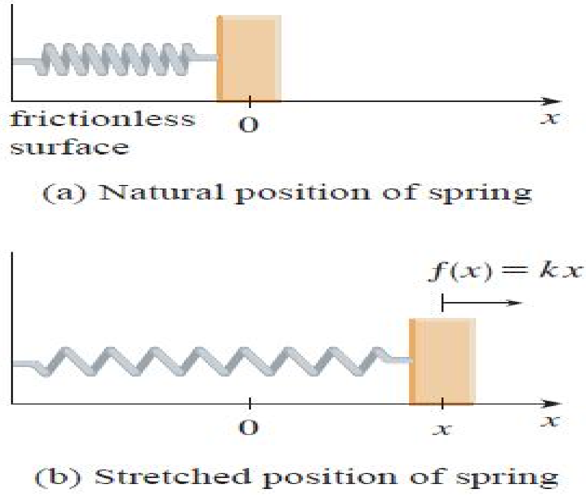
The force exerted on a mass by a spring is proportional to the displacement of the mass from equilibrium.

A spring has a *natural* length in the absence of being stretched or compressed. What Hooke's law says is that to stretch or compress a spring x units from this natural length requires a force

$$F(x) = kx.$$

The value k is called a *spring constant*.

Hooke's Law



frictionless surface
O
x
(a) Natural position of spring
$f(x) = kx$
O
x
x
(b) Stretched position of spring

Example

A force of 10 N is required to stretch a spring 25 cm from its equilibrium. Find the work done stretching the spring from its natural length of 5 cm to a length of 50 cm.

Find the spring constant : $F = kx$

$$10\,N = k\left(\frac{1}{4}\,m\right) \Rightarrow k = 40\,\frac{N}{m}$$

Our variable force is $F(x) = 40\frac{N}{m}\cdot x\,m = 40x\ N$

at natural length (5 cm) $x = 0\ m$

at 50 cm, $x = (0.5 - 0.05)\,m = 0.45\ m$

$$W = \int_0^{0.45} 40x \ N \ dx \ m = \int_0^{0.45} 40x \ dx \quad Nm$$

$$= 40 \frac{x^2}{2} \Big|_0^{0.45} \quad J$$

$$= 20 \left[(0.45)^2 - 0^2 \right] \quad J$$

$$= 20 (0.45)^2 \quad J$$

Example

A spring is stretched from its natural length of 1 foot to a length of 3 ft. The work done is 72 ft lb. Determine the spring constant k.

At 1 ft, displacement $x = 0$ ft

3 ft, " $x = 2$ ft

By Hooke's law $F(x) = kx$

$$W = 72 \text{ ft}\cdot\text{lb} = \int_0^2 \underbrace{kx}_{\text{lb}}\, \underbrace{dx}_{\text{ft}}$$

$$72 = k\frac{x^2}{2}\Big|_0^2 = k\left(\frac{2^2}{2} - \frac{0^2}{2}\right) = 2k$$

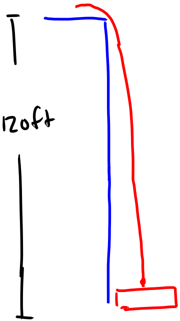
$$\implies k = 36.$$

The spring constant is $k = 36\ \frac{\text{lb}}{\text{ft}}$.

Another Work Example

A chain weighing 2 lb/ft is used to haul a 200 lb girder to the top of a building 120 ft high. Find the work done lifting the girder to the top of the building. (Neglect the height of the girder—i.e. assume it is lifted 120 ft.)

Total Work = Work lifting girder + work lifting chain.



Let work lifting girder be W_g

work lifting chain be W_c

$$W_g = 200\,\text{lb} \cdot 120\,\text{ft} = 24{,}000\ \text{ft}\cdot\text{lb}$$

We'll set up the process

to find W_c and

finish on 6/9/15