June 9 Math 2254 sec 001 Summer 2015

Section 6.6 Work

Work with Variable Force

Suppose we wish to move an object along the x-axis from $x = a$ to $x = b$ applying a variable force $F(x)$ in the direction of the motion. The work W is

$$W = \int_a^b F(x)\,dx$$

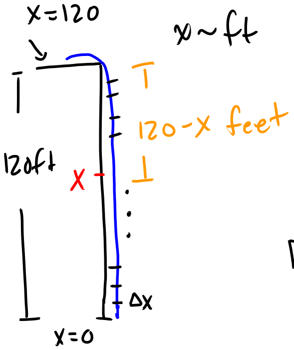
If F is in pounds, then x is in feet and W is in foot-pounds. If F is in Newtons, then x is in meters and W is in Newton-meters also called Joules.

Work Example

A chain weighing 2 lb/ft is used to haul a 200 lb girder to the top of a building 120 ft high. Find the work done lifting the girder to the top of the building. (Neglect the height of the girder—i.e. assume it is lifted 120 ft.)

We determined yesterday that the total work W is the sum of the work lifting the girder W_g and the work lifting the chain W_c. We also calculated

$$W_g = 200 \text{ lb} \times \ 120 \text{ ft} = 24000 \text{ ft lb}$$



Consider one piece of the chain of length Δx at some height x $(0 \le x \le 120)$.

Force for this piece:

$$\text{Force} = \text{weight} = 2\,\frac{\text{lb}}{\text{ft}} \cdot \Delta x \text{ ft}$$

$$F_{piece} = 2\,\Delta x \text{ lb}$$

Distance for this piece:

$$D_{piece} = (120 - x) \text{ ft}$$

The work for this piece of Chain is

$$W_{piece} = 2\Delta x \text{ lb } (120 - x) \text{ ft}$$

$$= 2(120 - x)\,\Delta x \text{ ft lb}$$

Summing these from $x = 0$ to $x = 120$ in the limit as $\Delta x \to 0$

$$W_c = \int_0^{120} 2(120 - x)\,dx \quad \text{ft} \cdot \text{lb}$$

$$= 2\left[120x - \frac{x^2}{2}\right|_0^{120} \ ft \cdot lb$$

$$= 2\left[120 \cdot 120 - \frac{120^2}{2} - 0\right] \ ft \cdot lb$$

$$= 2(120)^2\left(1 - \frac{1}{2}\right) \ ft \cdot lb$$

$$= 2(120)^2\left(\frac{1}{2}\right) \ ft \cdot lb$$

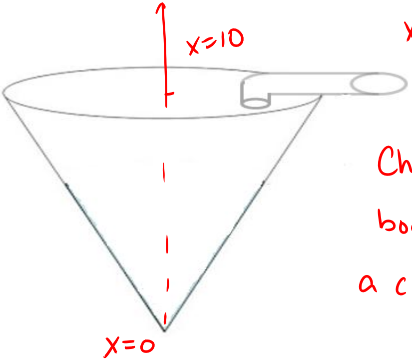
$$= 120^2 \ ft \cdot lb \qquad = 14400 \ ft \cdot lb$$

The total work lifting girder and chain is

$$W = 38400 \text{ ft} \cdot \text{lb}$$

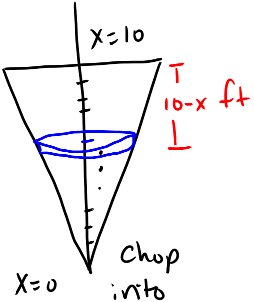
A Final Work Example

A tank of water in the shape of an inverted right circular cone is to be drained by pumping fluid through an opening at the top. The height and base radius of the cone are $h = 10$ ft and $r = 5$ ft, respectively. If water weighs 62 lb/ft^3, determine the work done emptying the tank.



x ~ ft

Characterize a body of fluid at a certain height.



Chop into pieces of width Δx

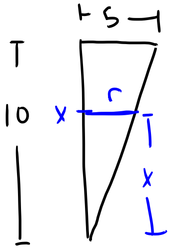
Take a slice of water of thickness Δx at a height x $(0 \le x \le 10)$

Distance for this slice

$$D_{slice} = (10-x)\text{ ft}$$

Force for this slice

$$F_{slice} = \text{weight} = \text{Volume} \times \text{density}$$



By similar triangles

$$\frac{5}{10} = \frac{r}{x} \Rightarrow r = \frac{1}{2}x \qquad r \sim ft$$

$$\Delta x \sim ft$$

$$V = \pi r^2 \Delta x \quad ft^3$$

$$= \pi \left(\frac{x}{2}\right)^2 \Delta x \quad ft^3$$

$$= \frac{\pi}{4} x^2 \Delta x \quad ft^3$$

$$F_{slice} = \frac{\pi}{4} x^2 \Delta x \ ft^3 \cdot 62 \frac{lb}{ft^3} = \frac{31}{2}\pi x^2 \Delta x \quad lb$$

Work for this slice

$$W_{slice} = \frac{31}{2}\pi x^2 \Delta x \text{ lb} \cdot (10-x) \text{ ft}$$

$$= \frac{31}{2}\pi x^2 (10-x)\,\Delta x \quad \text{ft} \cdot \text{lb}$$

Summing from $x=0$ to $x=10$ in the limit as $\Delta x \to 0$ gives the total work

$$W = \int_0^{10} \frac{31\pi}{2}\left(10x^2 - x^3\right) dx \quad \text{ft} \cdot \text{lb}$$

$$= \frac{31\pi}{2}\left[10\frac{x^3}{3} - \frac{x^4}{4}\right|_0^{10} \text{ ft}\cdot\text{lb}$$

$$= \frac{31\pi}{2}\left[10\,\frac{10^3}{3} - \frac{10^4}{4} - 0\right] \text{ ft}\cdot\text{lb}$$

$$= \frac{31\pi}{2}\,10^4\left(\frac{1}{3} - \frac{1}{4}\right) \text{ ft}\cdot\text{lb}$$

$$= \frac{31\pi}{2}\,10^4\left(\frac{4-3}{12}\right) \text{ ft lb}$$

$$= \frac{31\pi}{24}\cdot 10^4 \text{ ft}\cdot\text{lb}$$

Section 7.1: Integration by Parts

(The first of several new techniques for evaluating integrals)

Recall the product rule:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

Let's subtract gf' from both sides and consider an anti-derivative:

$$f(x)\,g'(x) = \frac{d}{dx}\left[f(x)\,g(x)\right] - g(x)\,f'(x)$$

$$\int f(x)\,g'(x)\,dx = \int \left(\frac{d}{dx}\left[f(x)\,g(x)\right] - g(x)\,f'(x)\right) dx$$

$$= \int \frac{d}{dx}\left[f(x)\,g(x)\right] dx - \int g(x)\,f'(x)\,dx$$

$$= f(x)\,g(x) - \int g(x)\,f'(x)\,dx$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Let $u = f(x)$ and $v = g(x)$

So $du = f'(x)dx$ and $dv = g'(x)\,dx$

$f(x)g'(x)dx = u\,dv$, $g(x)f'(x) = v\,du$

$$\int u\,dv = uv - \int v\,du$$

$\int u\,dv = uv - \int v\,du$

Evaluate
$$\int x\cos x\,dx$$

Need to choose u and dv

Set $u = x$ then $dv = \cos x\,dx$

$du = dx$ and $v = \sin x$

$$\int x\cos x\,dx = \underbrace{x}_{u}\underbrace{\sin x}_{v} - \int \underbrace{\sin x}_{v}\,\underbrace{dx}_{du}$$

$$= x\sin x - (-\cos x) + C$$

$$= x\sin x + \cos x + C$$