Mar. 10 Math 2254H sec 015H Spring 2015

Section 5.4: Areas in Polar Coordinates

The area bounded by the polar curve $r = f(\theta)$ over the wedge $\alpha \le \theta \le \beta$ is given by

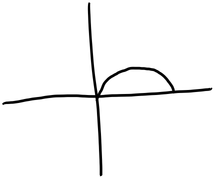
$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

▸ Polar Curve Java Applet

Example

Find the area of the region enclosed in one petal of the four petal rose $r = 2\cos 2\theta$.

Find when $r=0$ the first time



$2\cos(2\theta)=0$

$\cos(2\theta)=0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$

$\frac{1}{2}$ Petal is mapped out in the interval

$0 \le \theta \le \frac{\pi}{4}$

$$\frac{1}{2}A = \int_0^{\pi/4} \frac{1}{2} r^2 \, d\theta \Rightarrow A = \int_0^{\pi/4} (2\cos 2\theta)^2 \, d\theta$$

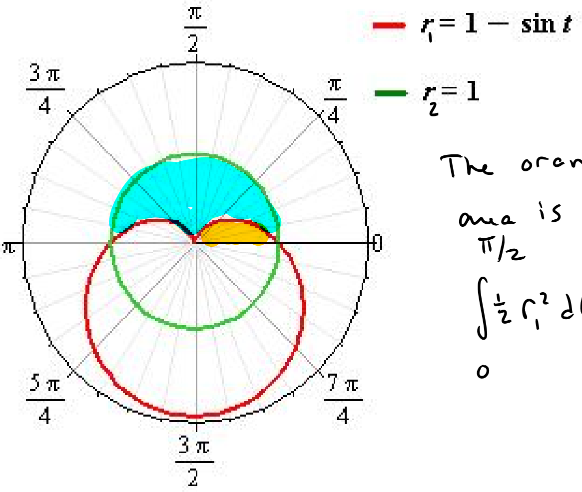
$$A = \int_0^{\pi/4} 4\cos^2 2\theta \, d\theta$$

$$= 4 \int_0^{\pi/4} \left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= 2 \left[\theta + \frac{1}{4}\sin 4\theta \right|_0^{\pi/4}$$

$$= 2\left[\frac{\pi}{4} + \frac{1}{4}\sin\pi - 0 - 0\right] = \frac{\pi}{2}$$

Find the area inside the circle $r = 1$ and outside of the cardioid $r = 1 - \sin\theta$.



The orange area is

$$\int_0^{\pi/2} \frac{1}{2} r_1^2 \, d\theta$$

$$\frac{1}{2}\int_0^{\pi/2} r_1^2\, d\theta = \frac{1}{2}\int_0^{\pi/2} (1-\sin\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(1 - 2\sin\theta + \sin^2\theta\right)\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(1 - 2\sin\theta + \frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\,\right|_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{3}{2}\cdot\frac{\pi}{2} + 2\cdot 0 - 0 - (0 + 2\cdot 1 - 0)\right]$$

$$= \frac{3\pi}{8} - 1$$

The area sought is

$$A = \frac{\pi}{2} - 2\left(\frac{3\pi}{8} - 1\right) = \frac{2\pi}{4} - \frac{3\pi}{4} + 2$$

$$= 2 - \frac{\pi}{4}$$