March 10 Math 2306 sec 58 Spring 2016

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Some Motivating Examples

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

We made a guess that $y_p = Ax + B$, a line, since the right hand side is a line. We found that this worked with $A = 2$ and $B = 9/4$ giving the particular solution $y_p = 2x + 9/4$.

$$y'' - 4y' + 4y = 6e^{3x}$$

We made a guess that $y_p = Ae^{3x}$ which matches the basic form of the right hand side. We found that this worked with $A = 6$ giving the particular solution $y_p = 6e^{3x}$.

Make the form general

$$y'' - 4y' + 4y = 16x^2$$

With this one, we tried setting $y_p = Ax^2$, but we couldn't solve the equation for any value of A.

We corrected it to the form $y_p = Ax^2 + Bx + C$ and found that this does solve the equation if $A = 4$, $B = 8$ and $C = 6$. So we found the particular solution $y_p = 4x^2 + 8x + 6$.

General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

Here, we have tried to find a particular solution using the guess

$$y_p = A\sin(2x).$$

The resulting equation upon substitution became

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

Unfortunately, this would require $A = -5 = 0$ which is always false! Hence we seek to accomodate the presense of the cosine by making the new guess

$$y_p = A\sin(2x) + B\cos(2x).$$

$$y_p' = 2A\cos(2x) - 2B\sin(2x)$$

$$y_p'' = -4A\sin(2x) - 4B\cos(2x)$$

we need

$$y_p'' - y_p' = 20\sin(2x)$$

$$-4A\sin(2x) - 4B\cos(2x) - \big(2A\cos(2x) - 2B\sin(2x)\big) = 20\sin(2x)$$

$$(\underline{-4A+2B})\sin(2x) + (\underline{\underline{-4B-2A}})\cos(2x) = \underline{20}\sin(2x) + \underline{\underline{0}}\cos(2x)$$

$$-4A + 2B = 20$$

$$-2A - 4B = 0 \quad \text{mult by } -2$$

$$-4A + 2B = 20$$

$$4A + 8B = 0$$

add

$$10B = 20$$

$$B = 2$$

From $-2A - 4B = 0$

$$2A = -4B \implies A = -2B$$

$$= -2(2) = -4$$

$$\boxed{\text{So } y_p = -4\sin(2x) + 2\cos(2x)}$$

Examples of Forms of y_p based on g (Trial Guesses)

(a) $g(x) = 1$ (or really any nonzero constant)

0 degree polynomial $\quad y_p = A$

(b) $g(x) = x - 7$

1^{st} degree poly. $\quad y_p = Ax + B$

(c) $g(x) = 5x$

$y_p = Ax + B$

(d) $g(x) = 3x^3 - 5$

3^{rd} degree poly $\quad y_p = Ax^3 + Bx^2 + Cx + D$

More Trial Guesses

(e) $g(x) = xe^{3x}$ product of 1^{st} degree poly and e^{3x}

$y_p = (Ax+B)e^{3x}$

(f) $g(x) = \cos(7x)$

$y_p = A\cos(7x) + B\sin(7x)$

(g) $g(x) = \sin(2x) - \cos(4x)$

$y_p = A\sin(2x) + B\cos(2x) + C\cos(4x) + D\sin(4x)$

(h) $g(x) = x^2\sin(3x)$ Quadratic times Sines / Cosines

$y_p = (Ax^2+Bx+C)\sin(3x) + (Dx^2+Ex+F)\cos(3x)$

Still More Trial Guesses

(i) $g(x) = e^x \cos(2x)$ $\quad y_p = Ae^x \cos(2x) + Be^x \sin(2x)$

(j) $g(x) = x^3 e^{8x}$ $\quad y_p = (Ax^3 + Bx^2 + Cx + D)e^{8x}$

(k) $g(x) = xe^{-x}\sin(\pi x)$

$$y_p = (Ax+B)e^{-x}\sin(\pi x) + (Cx+D)e^{-x}\cos(\pi x)$$

The Superposition Principle

$$y'' - y' = 20\sin(2x) + 4e^{-5x}$$

We can consider two subproblems

$y'' - y' = 20\sin(2x)$ solved by y_{p_1}

and $y'' - y' = 4e^{-5x}$ solved by y_{p_2}

We found that $y_{p_1} = -4\sin(2x) + 2\cos(2x)$

To find y_{p_2}, guess $y_{p_2} = Ae^{-5x}$

$$y_{p_2}' = -5Ae^{-5x}$$

$$y_{p_2}'' = 25Ae^{-5x}$$

$$y_{p_2}'' - y_{p_2}' = 4e^{-5x}$$

$$25Ae^{-5x} - (-5Ae^{-5x}) = 4e^{-5x}$$

$$30Ae^{-5x} = 4e^{-5x}$$

This requires $A = \frac{4}{30} = \frac{2}{15}$

So $y_{p_2} = \frac{2}{15} e^{-5x}$

By super position, $y_p = y_{p_1} + y_{p_2}$

$$y_p = -4\sin(2x) + 2\cos(2x) + \frac{2}{15} e^{-5x}$$

A Glitch!

$$y'' - y' = 3e^x$$

let's guess that $y_p = Ae^x$

$y_p' = y_p'' = Ae^x$

we need $y_p'' - y_p' = 3e^x$

$$Ae^x - Ae^x = 3e^x$$

$$0Ae^x = 3e^x$$

$0 = 3$ is false for all real A!

Consider the associated homogeneous eqn

$$y'' - y' = 0 \qquad \text{Ch. eqn.} \quad m^2 - m = 0$$

$$m(m-1) = 0$$

$$y_1 = e^{0x} = 1, \quad y_2 = e^{x} \qquad m = 0 \ \text{ or } \ m = 1$$

We'll consider cases

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply that term by x^n, where n is the smallest positive integer that eliminate the duplication.

Case II Example

Find the general solution.

$$y'' - 2y' + y = -4e^x$$

Find y_c: $\quad y'' - 2y' + y = 0$

Characteristic eqn. $\quad m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1$ repeated root

$y_1 = e^x, \quad y_2 = xe^x$

$y_c = c_1 e^x + c_2 x e^x$

Find y_p: Initial guess $y_p = Ae^x$

won't work, this is part of y_c

Try $y_p = Axe^x$ still part of y_c

Try $y_p = Ax^2e^x$

$y_p' = Ax^2e^x + 2Axe^x$

$y_p'' = Ax^2e^x + 2Axe^x + 2Axe^x + 2Ae^x$

$$y_p'' - 2y_p' + y_p = -4e^x$$

$$Ax^2e^x + 4Axe^x + 2Ae^x - 2\left(Ax^2e^x + 2Axe^x\right) + Ax^2e^x = -4e^x$$

$$(A - 2A + A)x^2e^x + (4A - 4A)xe^x + 2Ae^x = -4e^x$$

$$2Ae^x = -4e^x \implies A = -2$$

So $y_p = -2x^2e^x$

Gen. Soln.

$$y = y_c + y_p$$

$$y = c_1 e^x + c_2 x e^x - 2x^2 e^x$$