March 16 Math 2306 sec 59 Spring 2016

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

The Method of Undetermined Coefficients

The idea here is to assume a form for the particular solution based on the form of the right hand side $g(x)$. We have to keep a few things in mind.

- A coefficient such as m in e^{mx} or in $\cos(mx)$ should be left as it is. Only outer coefficients are to be determined.
- View functions in general terms—i.e. think of "x^3" as a cubic polynomial, and "xe^x" as a line times e^x.
- Where sines go, cosines must follow and vice versa.
- In general, every type of term that can arise in a derivative should be accounted for.

Example

Find the general solution of the ODE.

$$y'' - y' = 4xe^{2x}$$

Find y_c: It solves $y'' - y' = 0$

Characteristic Eqn $m^2 - m = 0$

$m(m-1) = 0 \Rightarrow$ $m = 0$ or $m = 1$

$y_1 = e^{0x} = 1$, $y_2 = e^{1x} = e^x$

$y_c = c_1 y_1 + c_2 y_2 = c_1 + c_2 e^x$

Find y_p where $y_p'' - y_p' = 4xe^{2x}$

guess $y_p = (Ax+B)e^{2x}$

$$y_p' = Ae^{2x} + 2(Ax+B)e^{2x}$$

$$y_p'' = 2Ae^{2x} + 2Ae^{2x} + 4(Ax+B)e^{2x}$$

$$y_p'' - y_p' = 4xe^{2x}$$

$$4Ae^{2x} + 4(Ax+B)e^{2x} - \left(Ae^{2x} + 2(Ax+B)e^{2x}\right) = 4xe^{2x}$$

$$3Ae^{2x} + 2(Ax+B)e^{2x} = 4xe^{2x}$$

$$3Ae^{2x} + 2Axe^{2x} + 2Be^{2x} = 4xe^{2x}$$

$$2Axe^{2x} + (3A+2B)e^{2x} = 4xe^{2x} + 0e^{2x}$$

Match like terms

$$2A = 4 \implies A = 2$$

$$3A + 2B = 0$$

$$2B = -3A = -3\cdot 2 = -6 \implies B = -3$$

so $y_p = (2x-3)e^{2x}$

The general solution $y = y_c + y_p$

so
$$y = C_1 + C_2 e^{x} + (2x-3)e^{2x}.$$

A Glitch!

$$y'' - y' = 3e^x$$

From the last example $y_c = c_1 + c_2 e^x$.

For y_p, let's guess $y_p = Ae^x$

$$y_p' = y_p'' = Ae^x$$

We need $y_p'' - y_p' = 3e^x$

$$Ae^x - Ae^x = 3e^x \Rightarrow 0 = 3$$

This is always false!

Since Ae^x solves the homogeneous equation, this initial guess must be modified,

We'll consider cases

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply that term by x^n, where n is the smallest positive integer that eliminates the duplication.

Case II Examples

Solve the ODE

$$y'' - 2y' + y = -4e^x$$

Find y_c : $y'' - 2y' + y = 0$

Characteristic Eqn $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$ $m = \underline{1}$ repeated root

$y_1 = e^x$, $y_2 = xe^x$

$y_c = c_1 y_1 + c_2 y_2 = c_1 e^x + c_2 x e^x$

Find y_p where $y_p'' - 2y_p' + y_p = -4e^x$

Guess $y_p = Ae^x$

This duplicates the y_1 in y_c. It won't work.

Try $y_p = Axe^x$ Still won't work.

Try $y_p = Ax^2e^x$ this will work.

$y_p' = Ax^2e^x + 2Axe^x$

$y_p'' = Ax^2e^x + 2Axe^x + 2Axe^x + 2Ae^x$

$$y_p'' - 2y_p' + y_p = -4e^x$$

$$Ax^2e^x + 4Axe^x + 2Ae^x - 2(Ax^2e^x + 2Axe^x) + Ax^2e^x = -4e^x$$

$$(A - 2A + A)x^2e^x + (4A - 4A)xe^x + 2Ae^x = -4e^x$$

$$\Rightarrow \quad 2Ae^x = -4e^x$$

which holds provided $A = -2$

So $y_p = -2x^2 e^x$

The general solution $y = y_c + y_p$

So

$$y = c_1 e^x + c_2 x e^x - 2x^2 e^x$$

Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

We'll consider subproblems

$y'' - 4y' + 4y = \sin(4x)$ solved by y_{p_1}

and $y'' - 4y' + 4y = xe^{2x}$ solved by y_{p_2}

Consider y_c: $y'' - 4y' + 4y = 0$

Chr. eqn $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0$

$m = 2$ repeated

$$y_1 = e^{2x}, \quad y_2 = xe^{2x}$$

For y_{p_1} guess $y_{p_1} = A\sin(4x) + B\cos(4x)$

No duplication of y_c. This will work.

For y_{p_2} guess

$$y_{p_2} = (Cx + D)e^{2x} = Cxe^{2x} + De^{2x}$$

Both terms duplicate y_c - modify

We can try mult. by x

$$y_{p_2} = (Cx+D)x e^{2x} = Cx^2 e^{2x} + \underline{\underline{Dxe^{2x}}}$$

this term still matches part of the fundamental solution set.

Try $$y_{p_2} = (Cx+D)x^2 e^{2x} = (Cx^3 + Dx^2)e^{2x}$$

This works.

So for the whole problem, $y_p = y_{p_1} + y_{p_2}$

$$y_p = A\sin(4x) + B\cos(4x) + \left(Cx^3 + Dx^2\right)e^{2x}$$

Find the form of the particular soluition

$$y''' - y'' + y' - y = \cos x + x^4$$

Consider the homogeneous eqn

$$y''' - y'' + y' - y = 0$$

Charc. Eqn $\quad m^3 - m^2 + m - 1 = 0$

Factor by grouping

$$m^2(m-1) + (m-1) = 0$$

$$(m-1)(m^2+1)=0$$

$$m=1 \quad \text{or} \quad m=\pm i$$

$\alpha \pm i\beta$

so

$\alpha=0, \ \beta=1$

So $\quad y_1=e^{x}, \quad y_2=e^{0x}\cos(1x)=\cos x$

and $\quad y_3=e^{0x}\sin(1x)=\sin x$

Let y_{p_1} solve $\quad y'''-y''+y'-y=\cos x$

and y_{p_2} solve $\quad y'''-y''+y'-y=x^4$

For y_{p_1} guess $y_{p_1} = A\cos x + B\sin x$

won't work - modify it

Try $y_{p_1} = (A\cos x + B\sin x)\,x$

$= Ax\cos x + Bx\sin x$ This works

For y_{p_2} try $y_{p_2} = Cx^4 + Dx^3 + Ex^2 + Fx + G$

No duplication - works as written.

So

$$y_p = y_{p_1} + y_{p_2}$$

$$= Ax\cos x + Bx\sin x + Cx^4 + Dx^3 + Ex^2 + Fx + G$$