March 10 Math 2335 sec 51 Spring 2016

Section 4.5 (& 4.6): Chebyshev Polynomials

We were considering the choice of nodes used for polynomial interpolation and the error contribution resulting from the way the nodes are distributed. For the moment, we restrict attention to data $\{(x_i, y_i) \,|\, -1 \le x_i \le 1\}$. We defined the Chebyshev polynomials:

Definition: For an integer $n \ge 0$ define the function

$$T_n(x) = \cos\left(n\,\cos^{-1}(x)\right), \quad -1 \le x \le 1.$$

It can be shown that T_n is a polynomial of degree n. It's called the

Chebyshev Polynomial of degree n.

Recursion Relation

It is readily verified by direct computation that $T_0(x) = 1$ and $T_1(x) = x$.

Then for $n \geq 1$, some straightforward applications of the *sum of angles* formula for the cosine yields the recursion relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

From this, we can generate a list

$$\begin{aligned} T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \\ T_4(x) &= 8x^4 - 8x^2 + 1 \\ T_5(x) &= 16x^5 - 20x^3 + 5x \\ &\vdots \end{aligned}$$

The Modified Chebyshev Polynomials

It can be shown that

$$T_n(x) = 2^{n-1}x^n + \text{lower power terms.}$$

We define the monic polynomials called the modified Chebyshev polynomials by

$$\tilde{T}_n(x) = \frac{1}{2^{n-1}} T_n(x).$$

Note that this scaling does not change the location of the roots. That is, $\tilde{T}_n$ has the same zeros as T_n.

Minimum Size Property

The property most useful to our purpose of minimizing error is given in the following theorem:

Theorem: Let $n \geq 1$ be an integer. Of all monic polynomials on the interval $[-1, 1]$, the one with the smallest maximum value is the modified Chebyshev polynomial $\tilde{T}_n(x)$. Moreover

$$|\tilde{T}_n(x)| \leq \frac{1}{2^{n-1}} \quad \text{for all} \quad -1 \leq x \leq 1.$$

This result suggests that whenever possible, we choose the polynomial $\Psi_n(x)$ in our error theorem to be the modified Chebyshev polynomial $\tilde{T}_{n+1}(x)$. This only requires the nodes to be the roots of T_{n+1}.

Chebyshev Nodes

To interpolate $f(x)$ on the interval $[-1, 1]$ by $P_n(x)$, the error is minimized by choosing the Chebyshev nodes (roots of $T_{n+1}(x)$)

$$x_j = \cos\left(\frac{(2j+1)\pi}{2(n+1)}\right), \quad j = 0, 1, \ldots n.$$

The resulting error bound is

$$|f(x) - P_n(x)| \leq \frac{L}{2^n}, \quad \text{where} \quad L = \max_{-1 \leq x \leq 1} \left| \frac{f^{(n+1)}(x)}{(n+1)!} \right|$$

Error Reduction

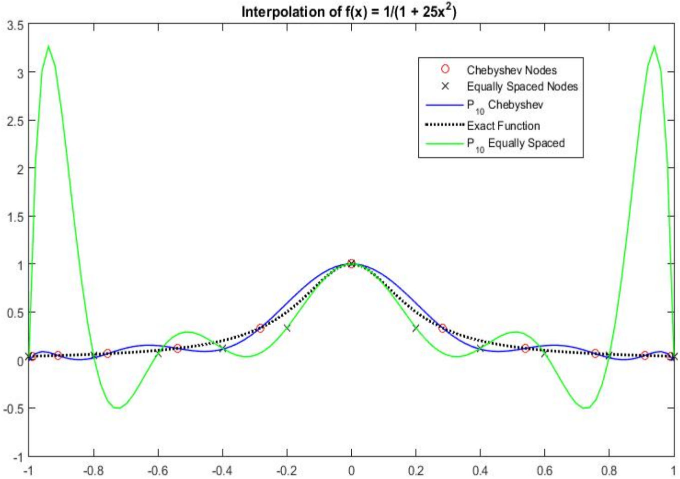

Figure: The maximum error using equally spaced nodes is 3.2214 whereas the maximum error using the Chebyshev nodes is 0.1090.

Example

Consider using P_4 to approximate the function $f(x) = \ln(2 + x)$ over the interval $[-1, 1]$. Determine the error minimizing nodes, and find a bound on the error $|f(x) - P_4(x)|$ based on this choice of nodes.

The optimal nodes are the roots of T_5.

$$x_j = \cos\left(\frac{(2j+1)\pi}{2\cdot 5}\right) \quad j = 0,1,2,3,4$$

$x_0 = \cos\left(\frac{\pi}{10}\right) \doteq 0.9511$, $\quad x_1 = \cos\left(\frac{3\pi}{10}\right) \doteq 0.5878$

$x_2 = \cos\left(\frac{5\pi}{10}\right) = 0$, $\quad x_3 = \cos\left(\frac{7\pi}{10}\right) \doteq -0.5878$,

$x_4 = \cos\left(\frac{9\pi}{10}\right) \doteq -0.9511$

We need a bound on $f^{(5)}(x)$ on $[-1,1]$.

$f(x) = \ln(2+x)$

$f'(x) = \dfrac{1}{2+x}$

$f''(x) = \dfrac{-1}{(2+x)^2}$

$f'''(x) = \dfrac{1 \cdot 2}{(2+x)^3}$

$f^{(4)}(x) = \dfrac{-1 \cdot 2 \cdot 3}{(2+x)^4}$

$f^{(5)}(x) = \dfrac{1 \cdot 2 \cdot 3 \cdot 4}{(2+x)^5}$ This is decreasing on $[-1,1]$.

So

$$L = \max_{-1 \le x \le 1} \left| \frac{f^{(5)}(x)}{5!} \right| = \frac{f^{(5)}(-1)}{5!} = \frac{\frac{4!}{(2-1)^5}}{5!}$$

$$= \frac{4!}{5!} = \frac{1}{5}$$

$$\left| f(x) - P_4(x) \right| \leq \frac{L}{2^4} = \frac{\frac{1}{5}}{16} = \frac{1}{80}$$

Intervals Other than $[-1, 1]$

The Chebyshev nodes are in the interval $[-1, 1]$. What if a function $f(x)$ is to be approximated on an interval $[a, b]$ different from $[-1, 1]$?

Let $a \leq x \leq b$. Find a formula in the form of a line for a new variable t such that $-1 \leq t \leq 1$. (That is, write $t = mx + \cancel{b}$.)

When $x=a$, we want $t=-1$ and when $x=b$, we want $t=1$. So find the line through $(a,-1)$ and $(b,1)$.

Slope $m = \dfrac{1-(-1)}{b-a} = \dfrac{2}{b-a}$

$$t - (-1) = \frac{2}{b-a}(x-a)$$

$$\Rightarrow \quad t = \frac{2}{b-a}(x-a) - 1$$

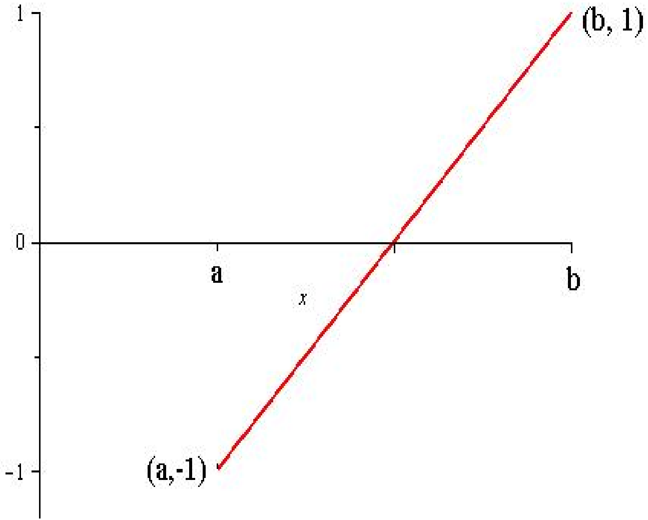

Figure: Line through points $(a, -1)$ and $(b, 1)$.

Intervals Other than $[-1, 1]$

Solve for x in terms of t from $\quad t = \dfrac{2}{b-a}(x-a)-1.$

$$\frac{2}{b-a}(x-a) = t+1 \quad \Rightarrow$$

$$x-a = \frac{b-a}{2}(t+1) \quad \Rightarrow$$

$$x = \frac{b-a}{2}(t+1) + a$$

Intervals Other than $[-1, 1]$

For $f(x)$ defined on $[a, b]$, set

$$t = \frac{2}{b-a}(x-a) - 1$$

Then $-1 \le t \le 1$, and we can use the Chebyshev nodes for t. We perform the interpolation for

$$g(t) = f\left(\underbrace{\frac{b-a}{2}(t+1) + a}_{x}\right)$$

so this is $f(x)$.

Intervals Other than $[-1, 1]$

$$\text{for} \quad g(t) = f\left(\frac{b-a}{2}(t+1) + a\right)$$

Show that $\quad g'(t) = \left(\frac{b-a}{2}\right) f'(x).$

Note $\quad \frac{dx}{dt} = \frac{b-a}{2}$

By the chain rule $\quad g'(t) = \frac{df}{dx}\frac{dx}{dt}$

$$= \frac{b-a}{2} f'(x)$$

Error Formula for Intervals Other than $[-1, 1]$

If (translated) Chebyshev nodes are used to approximate $f(x)$ on $[a, b]$ by the interpolating polynomial $P_n(x)$, then the error

$$|f(x) - P_n(x)| \leq \left(\frac{b-a}{2}\right)^{n+1} \frac{L}{2^n},$$

$$\text{where} \quad L = \max_{a \leq x \leq b} \left| \frac{f^{(n+1)}(x)}{(n+1)!} \right|$$

Example Finding Shifted Nodes

Let $1 \leq x \leq 2$. Determine the shifted Chebyshev nodes for use with P_3.

here $a=1$ and $b=2$ so

$$x = \frac{b-a}{2}(t+1) + a = \frac{1}{2}(t+1)+1 = \frac{1}{2}t + \frac{3}{2}$$

We need nodes t from T_4

$$t_j = \cos\left(\frac{(2j+1)\pi}{2\cdot 4}\right) \quad j = 0, 1, 2, 3$$

$$t_0 = \cos\left(\frac{\pi}{8}\right) \doteq 0.9239, \quad t_1 = \cos\left(\frac{3\pi}{8}\right) \doteq 0.3827$$

$$t_2 = \cos\left(\frac{5\pi}{8}\right) \doteq -0.3827, \quad t_3 = \cos\left(\frac{7\pi}{8}\right) \doteq -0.9239$$

$$x_0 = \frac{1}{2}t_0 + \frac{3}{2} \doteq 1.9619$$

$$x_1 = \frac{1}{2}t_1 + \frac{3}{2} \doteq 1.6913$$

$$x_2 = \frac{1}{2}t_2 + \frac{3}{2} \doteq 1.3087$$

$$x_3 = \frac{1}{2}t_3 + \frac{3}{2} \doteq 1.0381$$

Example Bounding Error on $[a, b]$

Let $f(x) = \tan^{-1}(x)$ on the interval $[0, 1]$. Suppose we wish to interpolate $f(x)$ using $P_2(x)$. Determine the translated Chebyshev nodes. And find a bound on the error when these nodes are used.

Here $a=0$, $b=1$ so $x = \frac{1-0}{2}(t+1) + 0 = \frac{1}{2}(t+1)$

We need the roots of T_3

$$t_j = \cos\left(\frac{(2j+1)\pi}{2\cdot 3}\right), \quad j = 0, 1, 2$$

$$t_0 = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad t_1 = \cos\left(\frac{3\pi}{6}\right) = 0, \quad t_2 = \cos\left(\frac{5\pi}{6}\right) = \frac{-\sqrt{3}}{2}$$

Example Continued...[1]

$x_0 = \frac{1}{2}(t_0+1) \doteq 0.9330$

$x_1 = \frac{1}{2}(t_1+1) = \frac{1}{2}$

$x_2 = \frac{1}{2}(t_2+1) \doteq 0.0670$

Note $L = \max_{0\le x\le 1} \left| \frac{f^{(3)}(x)}{3!} \right| = \frac{f^{(3)}(1)}{3!} = \frac{\frac{6-2}{(1+1)^3}}{3!} = \frac{\frac{4}{8}}{6}$

$= \frac{1}{12}$

[1] It's helpful to note that

$$\frac{d^3}{dx^3}\tan^{-1}x = \frac{6x^2-2}{(x^2+1)^3}$$

which is increasing on [0, 1].

So

$$|f(x) - P_2(x)| \le \left(\frac{1-0}{2}\right)^3 \frac{L}{2^2} = \left(\frac{1}{8}\right) \frac{\frac{1}{12}}{4}$$

$$= \frac{1}{8 \cdot 4 \cdot 12} = \frac{1}{384}$$