March 11 MATH 1112 sec. 2 Spring 2020

Fundamental Identities: Pythagorean, Sum, and Difference

Identities we already know:

Reciprocal: $$\csc(x) = \frac{1}{\sin(x)}, \quad \sec(x) = \frac{1}{\cos(x)}, \quad , \cot(x) = \frac{1}{\tan(x)},$$

$$\sin(x) = \frac{1}{\csc(x)}, \quad \cos(x) = \frac{1}{\sec(x)}, \quad , \tan(x) = \frac{1}{\cot(x)},$$

Quotient: $$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \cot(x) = \frac{\cos(x)}{\sin(x)}$$

Identities We Already Know

Cofunction: $$\cos(x) = \sin\left(\frac{\pi}{2} - x\right), \quad \sin(x) = \cos\left(\frac{\pi}{2} - x\right),$$

$$\csc(x) = \sec\left(\frac{\pi}{2} - x\right), \quad \sec(x) = \csc\left(\frac{\pi}{2} - x\right),$$

$$\cot(x) = \tan\left(\frac{\pi}{2} - x\right), \quad \tan(x) = \cot\left(\frac{\pi}{2} - x\right).$$

Periodicity: $$\sin(x + 2\pi) = \sin(x), \quad \cos(x + 2\pi) = \cos(x)$$

$$\csc(x + 2\pi) = \csc(x), \quad \sec(x + 2\pi) = \sec(x)$$

$$\tan(x + \pi) = \tan(x), \quad \cot(x + \pi) = \cot(x)$$

Symmetry: $$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \tan(-x) = -\tan(x)$$

$$\csc(-x) = -\csc(x), \quad \sec(-x) = \sec(x), \quad \cot(-x) = -\cot(x).$$

Pythagorean Identities

$$\sin^2(x) + \cos^2(x) = 1$$

$$\tan^2(x) + 1 = \sec^2(x), \quad \text{and}$$

$$1 + \cot^2(x) = \csc^2(x)$$

Commit these to memory

These can be arranged in various, useful ways.

e.g. $\cos^2 x = 1 - \sin^2 x$

Sum and Difference Identities

Cosine Identities:

(sum) $\cos(u+v) = \cos u \cos v - \sin u \sin v,$

(diff) $\cos(u-v) = \cos u \cos v + \sin u \sin v$

Sine Identities:

(sum) $\sin(u+v) = \sin u \cos v + \sin v \cos u$

(diff) $\sin(u-v) = \sin u \cos v - \sin v \cos u$

Tangent Identities:

(sum) $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v},$

(diff) $\tan(u-v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

These IDs will be provided on the next test

You don't have to memorize them, just know how to use them.

Evaluate

$$\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{3\pi}{10}\right)-\sin\left(\frac{\pi}{5}\right)\sin\left(\frac{3\pi}{10}\right)$$

$$= \cos\left(\frac{\pi}{5} + \frac{3\pi}{10}\right)$$

$$= \cos\left(\frac{\pi}{2}\right) = 0$$

* Product of cosines − product of sines

* $\frac{\pi}{5} + \frac{3\pi}{10} = \frac{2\pi}{10} + \frac{3\pi}{10} = \frac{5\pi}{10} = \frac{\pi}{2}$

Evaluate $\csc(\alpha + \beta)$ using the given information.

Given: $\tan\alpha = -2, \quad \frac{\pi}{2} < \alpha < \pi$ ← quad II

and $\sec\beta = \frac{4}{3}, \quad \frac{3\pi}{2} < \beta < 2\pi$ ← quad IV

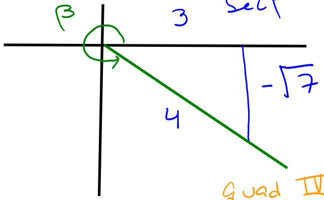
We'll start with diagrams for α and β.

$\tan\alpha = \frac{2}{-1}$

$\sec\beta = \frac{4}{3}$

* quad II

$x < 0, \; y > 0$



quad IV

$x > 0$

$y < 0$

Note $\sin\alpha = \frac{2}{\sqrt{5}}$

$\cos\alpha = \frac{-1}{\sqrt{5}}$

$\sin\beta = \frac{-\sqrt{7}}{4}$

$\cos\beta = \frac{3}{4}$

$$\csc(\alpha+\beta) = \frac{1}{\sin(\alpha+\beta)}$$ and

$$\sin(\alpha+\beta) = \sin\alpha \cos\beta + \sin\beta \cos\alpha$$

$$= \frac{2}{\sqrt{5}}\left(\frac{3}{4}\right) + \left(\frac{-\sqrt{7}}{4}\right)\left(\frac{-1}{\sqrt{5}}\right)$$

$$= \frac{6+\sqrt{7}}{4\sqrt{5}}$$

Finally, $$\csc(\alpha+\beta) = \frac{4\sqrt{5}}{6+\sqrt{7}}$$

Verifying Identities

Let me verify the identity

$$\csc(x) - \sin(x) = \cos(x)\cot(x)$$

I'll work with the left side, apply trig IDs and algebra, and try to arrive @ the right.

$$\csc x - \sin x = \frac{1}{\sin x} - \sin x$$ reciprocal ID

$$= \frac{1}{\sin x} - \frac{\sin^2 x}{\sin x}$$ algebra common denom.

$$= \frac{1 - \sin^2 x}{\sin x}$$ algebra add

$$= \frac{\cos^2 x}{\sin x}$$ Pythagorean ID

$$= \cos x \, \frac{\cos x}{\sin x}$$ algebra

$$= \cos x \, \cot x$$ quotient ID

That's it! We've shown that

$\csc x - \sin x$ is equivalent to

$\cos x \cot x$.

Verifying Identities

Some things to note:

- Verifying an identity is **NOT** solving an equation.
- We do not ”do the same thing” to both sides.
- We do not assume the statement is true. We SHOW it!
- Pick one side, and apply identities to it. The goal is to transform it to the other side.
- Usually try to work with the *most complicated* side. (It’s usually easier to simplify a complicated expression than to complicate a simpler one!)
- Sometimes it helps to write everything in terms of sines and cosines—not always, but often.

Verify $\frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x}$

It is always legitimate to multiply by 1 or to add zero.

This example uses a common technique: multiply by 1, and make use of difference of squares.

Recall: $(a-b)(a+b) = a^2 - b^2$

Starting with the left side

$$\frac{\sin x}{1-\cos x} = \left(\frac{\sin x}{1-\cos x}\right)\left(\frac{1+\cos x}{1+\cos x}\right)$$

this is multiplying by 1

$$= \frac{\sin x\,(1+\cos x)}{1-\cos^2 x}$$ ← difference of squares

$$= \frac{\sin x\,(1+\cos x)}{\sin^2 x}$$ pythagorean ID

$$= \frac{\cancel{\sin x}\,(1+\cos x)}{\cancel{\sin x}\,\sin x}$$ algebra

$$= \frac{1+\cos x}{\sin x}$$ algebra

Done!

Verify $\dfrac{\sin(x-y)}{\cos x \cos y} = \tan x - \tan y$

We'll use a difference of angles formula

$$\sin(x-y) = \sin x \cos y - \sin y \cos x$$

From the left side

$$\frac{\sin(x-y)}{\cos x \cos y} = \frac{\sin x \cos y - \sin y \cos x}{\cos x \cos y}$$

$$= \frac{\sin x \cos y}{\cos x \cos y} - \frac{\sin y \cos x}{\cos x \cos y}$$

$$= \frac{\sin x \cancel{\cos y}}{\cos x \cancel{\cos y}} - \frac{\sin y \cancel{\cos x}}{\cancel{\cos x} \cos y}$$

$$= \frac{\sin x}{\cos x} - \frac{\sin y}{\cos y}$$

$$= \tan x - \tan y$$

Done!