March 11 Math 3260 sec. 51 Spring 2020

Section 4.4: Coordinate Systems

We begin with a theorem about uniqueness of linear combinations (of linearly independent vectors).

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each vector $\mathbf{x}$ in V, there is a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n.$$

Suppose we had two representations for a vector $\vec{x}$.

$$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n \quad \text{and}$$

$$\vec{x} = a_1\vec{b}_1 + a_2\vec{b}_2 + \cdots + a_n\vec{b}_n$$

The theorem says the c's must be the same as the a's. Let's subtract the bottom line from the top - note $\vec{x} - \vec{x} = \vec{0}$.

$$\vec{0} = (c_1 - a_1)\vec{b}_1 + (c_2 - a_2)\vec{b}_2 + \cdots + (c_n - a_n)\vec{b}_n$$

* The vectors $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ are linearly independent!

$$\Longrightarrow \quad c_1 - a_1 = 0 \qquad c_2 - a_2 = 0, \ldots, c_n - a_n = 0$$

$$\Longrightarrow \quad a_1 = c_1, \quad a_2 = c_2, \ldots, \quad a_n = c_n$$

That is, there is only <u>one</u> set of coefficients for $\vec{x}$.

Coordinate Vectors

Definition: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an **ordered** basis of the vector space V. For each $\mathbf{x}$ in V we define the **coordinate vector of x relative to the basis** $\mathcal{B}$ to be the unique vector $(c_1, \ldots, c_n)$ in $\mathbb{R}^n$ where these entries are the weights $\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n$.

We'll use the notation

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = [\mathbf{x}]_{\mathcal{B}}.$$

* No matter what kind of vector $\vec{x}$ is, $[\vec{x}]_{\mathcal{B}}$ is a vector in $\mathbb{R}^n$.

Example

Let $\mathcal{B} = \{1, t, t^2, t^3\}$ (in that order) in $\mathbb{P}_3$. Determine $[\mathbf{p}]_{\mathcal{B}}$ where

(a) $\mathbf{p}(t) = 3 - 4t^2 + 6t^3$ $= (3)\,1 + (0)\,t + (-4)\,t^2 + (6)\,t^3$

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ -4 \\ 6 \end{bmatrix}$$

a vector in $\mathbb{R}^4$

(b) $\mathbf{p}(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

again, $[\vec{p}]_{\mathcal{B}}$ is in $\mathbb{R}^4$

Example

Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ if $\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2$.

We can set up a vector equation,

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

In matrix format

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

This can be solved in various ways (e.g. rref, matrix inverse, Crammer's rule).

Using a matrix inverse: $\det\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = 2+1 = 3$

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}\vec{x} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

So $\vec{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ has coordinate vector $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ for basis $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

Coordinates in $\mathbb{R}^n$

Note from this example that $\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$ where $P_{\mathcal{B}}$ is the matrix $[\mathbf{b}_1\ \mathbf{b}_2]$. The matrix $P_{\mathcal{B}}$ is called the **change of coordinates matrix** for the basis $\mathcal{B}$ (or from the basis $\mathcal{B}$ to the standard basis).

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis of $\mathbb{R}^n$. Then the change of coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the linear transformation defined by

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$$

where the matrix

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n].$$

Example

Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. Determine the matrix $P_{\mathcal{B}}$ and its inverse.

From before

$$P_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \qquad P_{\mathcal{B}}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

Use this to find

(a) the coordinate vector of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

* Note $[\vec{b}_1]_{\mathcal{B}} = \vec{e}_1$

(b) the coordinate vector of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Note $[\vec{b}_2]_{\mathcal{B}} = \vec{e}_2$

(c) a vector **x** whose coordinate vector is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\vec{x} = P_{\mathcal{B}} [\vec{x}]_{\mathcal{B}}$, so

$$\vec{x} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

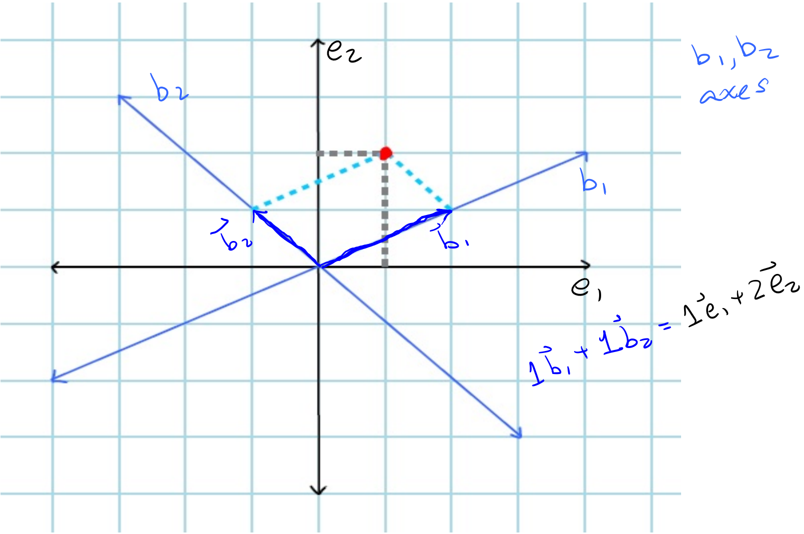

Figure: $\mathbb{R}^2$ shown using elementary basis $\{(1,0),(0,1)\}$ and with the alternative basis $\{(2,1),(-1,1)\}$.