Mar. 12 Math 2254H sec 015H Spring 2015

Section 11.1: Sequences

Definition: A **sequence** is an ordered list of numbers

$$a_1, a_2, a_3, \dots, a_n, \dots$$

Here, a_1 is called the *first term*, a_2, the *second term*, and in general a_n is called the n^{th} *term*.

A sequence can be considered as a function whose domain is the positive integers.

n is called the index

$$a_n = f(n)$$

We may also use the various notations

$$\{a_1, a_2, \dots\} \quad \text{or} \quad \{a_n\}, \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}.$$

Examples

Write the first four terms of the sequence defined by the indicated relation.

(a) $a_n = \dfrac{2n}{n+1} \quad n = 1, 2, \ldots$

$a_1 = \frac{2\cdot 1}{1+1} = 1$

$a_2 = \frac{2\cdot 2}{2+1} = \frac{4}{3}$

$a_3 = \frac{2\cdot 3}{3+1} = \frac{6}{4}$

$a_4 = \frac{2\cdot 4}{4+1} = \frac{8}{5}$

(b) $a_n = (-1)^n \quad n = 0, 1, \ldots$

$a_0 = (-1)^0 = 1$

$a_1 = (-1)^1 = -1$

$a_2 = (-1)^2 = 1$

$a_3 = (-1)^3 = -1$

A recursively defined sequence

The **Fibonacci sequence**, $\{f_n\}$ is defined by

$$f_0 = 1, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad \text{for} \quad n \geq 2.$$

Write the first 6 terms of this sequence.

$f_0 = 1$

$f_1 = 1$

$f_2 = f_1 + f_0 = 2$

$f_3 = f_2 + f_1 = 3$

$f_4 = f_3 + f_2 = 5$

$f_5 = f_4 + f_3 = 8$

Limits and Convergence

Definition: A sequence $\{a_n\}$ is said to be **convergent** with limit L provided

$$\lim_{n\to\infty} a_n = L.$$

A sequence that is not convergent is **divergent.**

Example: Determine if the sequence $a_n = \frac{2n}{n+1}$ is convergent or divergent. If convergent, determine the limit.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{n+1}$$

$$= \lim_{n\to\infty} \left(\frac{2n}{n+1}\right) \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{2}{1+\frac{1}{n}} = \frac{2}{1+0}$$

$$= 2$$

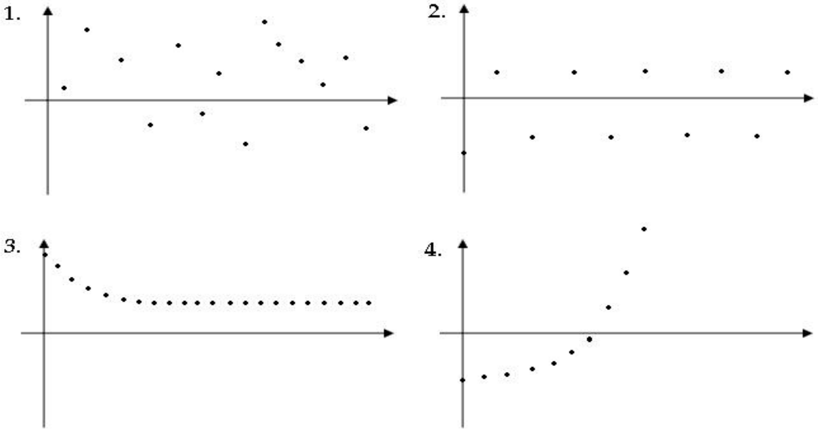

Figure: Plotted as points (n, a_n), sequence terms may jump around (1), oscillate back and forth between two or more values (2), converge to a limit (3), or become unbounded going to $+\infty$ or $-\infty$ (4).

Examples

Determine if the sequence is convergent or divergent. If convergent, find its limit.

(a) $a_n = (-1)^n \quad n \geq 0$

$a_0 = 1$

$a_1 = -1$

$a_2 = 1$

$a_3 = -1$

$\vdots$

$\lim_{n\to\infty} (-1)^n$ D.N.E.

divergent

(b) $b_n = 2^n \quad n \geq 0$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} 2^n = \infty$$

$b_0 = 2^0 = 1$

$b_1 = 2^1 = 2$

$b_2 = 2^2 = 4$

$\vdots$

divergent

(c) $c_n = \dfrac{1}{\ln n} \quad n \geq 2$

$$c_2 = \frac{1}{\ln 2}$$

$$c_3 = \frac{1}{\ln 3}$$

$$\vdots$$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{1}{\ln n}$$

$$= 0$$

Convergent w/ limit 0.

A Word of Caution about Derivatives

Remember that if $f'(x)$ exists, it is defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

If we have a function $F(n)$ whose domain is $1, 2, 3, \ldots$, then $n+h$ is not in the domain of F if h is not a positive integer.

That is, $F(n+h)$ DOES NOT MAKE SENSE IF h IS NOT AN INTEGER!!

So to be technically correct, we'll consider a function $f(x)$ whose variable x is a real number. We evaluate

$$\frac{df}{dx} \quad \text{and NOT} \quad \frac{df}{dn}.$$

Theorem

Theorem: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for each integer n, then $\lim_{n\to\infty} a_n = L$.

Example: Determine the limit of the sequence $a_n = \frac{\ln n}{n}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n} = \text{“}\frac{\infty}{\infty}\text{”} \quad \text{ind. form}$$

Let $f(x) = \frac{\ln x}{x}$ note $f(n) = a_n$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x} = \text{"}\frac{\infty}{\infty}\text{"} \quad \text{Use l'H rule}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

By our theorem, a_n converges and has limit 0.

Limit Laws for Sequences

Theorem: Suppose $\{a_n\}$ and $\{b_n\}$ are convergent to A and B, respectively, and let c be constant. Then

$$\begin{aligned}
\lim_{n\to\infty}(a_n \pm b_n) &= A \pm B \\
\lim_{n\to\infty} ca_n &= cA \\
\lim_{n\to\infty}(a_n b_n) &= AB \\
\lim_{n\to\infty} \frac{a_n}{b_n} &= \frac{A}{B} \quad \text{if} \quad b_n \neq 0, \quad B \neq 0 \\
\lim_{n\to\infty} [a_n]^p &= A^p \quad \text{if} \quad p > 0, \quad a_n \geq 0
\end{aligned}$$

Example

Use appropriate limit laws to determine the limit if it exists.

(a) $a_n = \sqrt{1 - \dfrac{1}{2^n}}$

$$\lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1 - 0 = 1$$

$1 - \frac{1}{2^n} \geq 0$ for all n so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{1 - \frac{1}{2^n}} = \sqrt{1}$$

(b) $b_n = \dfrac{n}{2n+1} - \dfrac{3n+1}{n+2}$

$$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \left(\frac{n}{2n+1}\right) \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{3n+1}{n+2} = \lim_{n\to\infty} \left(\frac{3n+1}{n+2}\right) \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3+\frac{1}{n}}{1+\frac{2}{n}} = 3$$

So $\displaystyle\lim_{n\to\infty} b_n = \frac{1}{2} - 3 = \frac{-5}{2}$

Squeeze Theorem

Theorem: Suppose $a_n \leq b_n \leq c_n$ for all $n \geq n_0$. If

$$\lim_{n\to\infty} a_n = L \quad \text{and} \quad \lim_{n\to\infty} c_n = L, \quad \text{then} \quad \lim_{n\to\infty} b_n = L.$$

$$-|a_n| \leq a_n \leq |a_n|$$

Corollary: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

The Squeeze Theorem

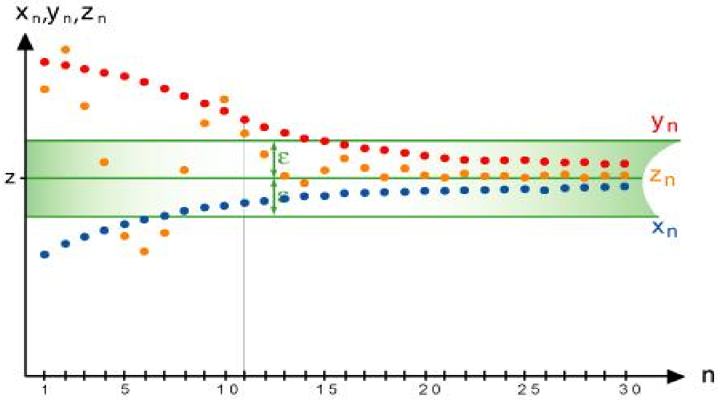

Figure: The sequence $\{z_n\}$ (orange) is *squeezed* between the sequences $\{x_n\}$ (blue) and $\{y_n\}$ (red) for all $n \geq 11$. Since $x_n \to z$ and $y_n \to z$, it is guaranteed that $z_n \to z$.

Factorials

For an integer $n \geq 1$ the expression $n!$, read *n factorial* is defined as the product of the first n integers. That is

$$n! = 1 \cdot 2 \cdot 3 \cdots n. \quad \text{Also} \quad 0! = 1.$$

Examples: Compute 4! and 7!.

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

$7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$

$= 4! \cdot 5 \cdot 6 \cdot 7 = 5040$

Show that $(n+1)! = n!(n+1)$.

$n! = 1 \cdot 2 \cdot 3 \cdots n$

$(n+1)! = \underbrace{1 \cdot 2 \cdot 3 \cdots n}_{n!} \cdot (n+1) = n!\,(n+1)$

Squeeze Theorem Example

Show that $0 \leq a_n \leq \frac{1}{n}$ and comment on the convergence or divergence of the sequence

$$a_n = \frac{n!}{n^n}.$$

$$a_n = \frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdot 4\cdots n}{n\cdot n\cdot n\cdot n\cdots n}$$

$$= \frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdot\frac{4}{n}\cdots\frac{n}{n}$$

$$\leq \frac{1}{n}\cdot 1\cdot 1\cdot 1\cdots 1 = \frac{1}{n}$$

Note, since $n!$ and n^n are positive

$$0 \le a_n$$

So $$0 \le a_n \le \frac{1}{n}.$$

$$\lim_{n\to\infty} 0 = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} = 0$$

By the squeeze theorem a_n converges

and $\lim_{n\to\infty} a_n = 0$.