March 13 Math 2306 sec. 57 Spring 2018

Section 11: Linear Mechanical Equations

Simple Harmonic Motion

We consider a flexible spring from which a mass is suspended. In the absence of any damping forces (e.g. friction, a dash pot, etc.), and free of any external driving forces, any initial displacement or velocity imparted will result in **free, undamped motion**–a.k.a. **simple harmonic motion.**

Harmonic Motion gif

Building an Equation: Hooke's Law

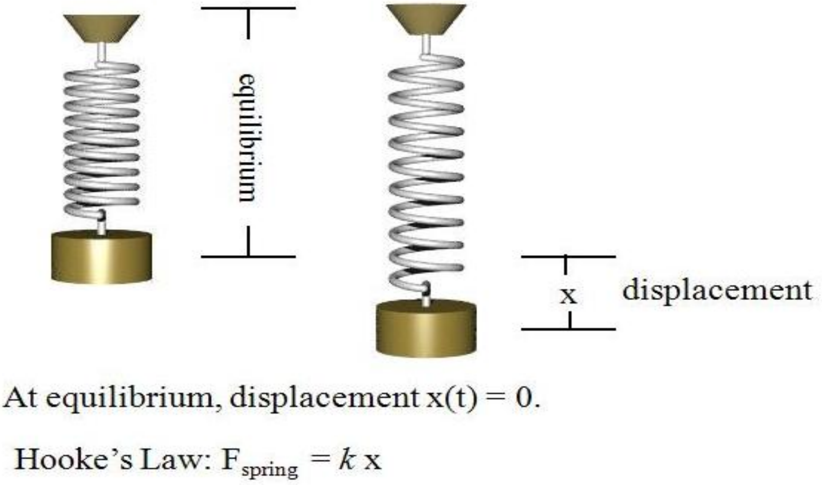

Figure: In the absence of any displacement, the system is at equilibrium. Displacement $x(t)$ is measured from equilibrium $x = 0$.

Building an Equation: Hooke's Law

Newton's Second Law: $F = ma$ Force = mass times acceleration

$$a = \frac{d^2x}{dt^2} \quad \Longrightarrow \quad F = m\frac{d^2x}{dt^2}$$

Hooke's Law: $F = kx$ Force exerted by the spring is proportional to displacement

The force imparted by the spring opposes the direction of motion.

$mx'' + kx = 0 \Rightarrow x'' + \frac{k}{m}x = 0$

$$m\frac{d^2x}{dt^2} = -kx \quad \Longrightarrow \quad x'' + \omega^2 x = 0 \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}$$

Convention We'll Use: Up will be positive ($x > 0$), and down will be negative ($x < 0$). This orientation is arbitrary and follows the convention in Trench.

Obtaining the Spring Constant (US Customary Units)

If an object with weight W pounds stretches a spring δx feet[1] from it's length with no mass attached, then by Hooke's law we compute the spring constant via the equation

$$W = k\delta x.$$

The units for k in this system of measure are lb/ft.

$$k = \frac{W \text{ lb}}{\delta x \text{ ft}} = \frac{W}{\delta x} \frac{\text{lb}}{\text{ft}}$$

[1] Note that $\delta x =$ w/ mass equilibrium - w/o mass equilibrium.

Obtaining the Spring Constant (US Customary Units)

Note also that Weight = mass $\times$ acceleration due to gravity. Hence if we know the weight of an object, we can obtain the mass via

$$W = mg.$$

We typically take the approximation $g = 32$ ft/sec^2. The units for mass are lb sec^2/ft which are called slugs.

$$m = \frac{W \text{ lb}}{g \text{ ft/sec}^2} = \frac{W}{g} \underbrace{\text{sec}^2 \frac{\text{lb}}{\text{ft}}}_{\text{slug}}$$

Obtaining the Spring Constant (SI Units)

In SI units, the weight would be expressed in Newtons (N). The appropriate units for displacement would be meters (m). In these units, the spring constant would have units of N/m. $k = \frac{W}{\delta x}$ $\frac{N}{m}$

It is customary to describe an object by its mass in kilograms. When we encounter such a description, we deduce the weight in Newtons

$$W = mg \quad \text{taking the approximation} \quad g = 9.8\,\text{m}/\text{sec}^2.$$

Obtaining the Spring Constant: *Displacment in Equilibrium*

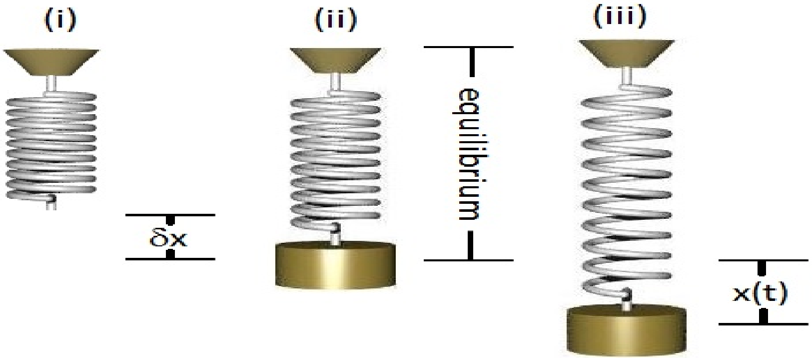

Figure: (i) Spring only *equilibrium*. (ii) Spring-mass system **equilibrium.** The difference δx will be called *displacement in equilibrium*. Our variable $x(t)$ will be displacement of the Spring-Mass system.

Obtaining the Spring Constant: *Displacment in Equilibrium*

If an object stretches a spring δx units from it's length (with no object attached), we may say that it stretches the spring δx units *in equilibrium*. Applying Hooke's law with the weight as force, we have

$$W = mg = k\delta x. \Rightarrow \frac{mg}{m\delta x} = \frac{k\delta x}{m\delta x} \Rightarrow \frac{g}{\delta x} = \frac{k}{m}$$

We observe that the value ω can be deduced from δx by

$$\omega^2 = \frac{k}{m} = \frac{g}{\delta x}.$$

Provided that values for δx and g are used in appropriate units, ω is in units of per second.

Simple Harmonic Motion

initial position

initial velocity

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \tag{1}$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The solution is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

called the **equation of motion.**

Caution: The phrase *equation of motion* is used differently by different authors. Some, including Trench, use this phrase to refer the ODE of which (1) would be the example here. Others use it to refer to the **solution** to the associated IVP.

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$[2]
- the circular (or angular) frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

[2]Various authors call f the natural frequency and others use this term for ω.

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ must be defined by

$$\sin\phi = \frac{x_0}{A}, \quad \text{with} \quad \cos\phi = \frac{x_1}{\omega A}.$$

Example

An object stretches a spring 6 inches in equilibrium. Assuming no driving force and no damping, set up the differential equation describing this system.

The ODE looks like $x'' + \omega^2 x = 0$.

We need ω^2. We're given displacement in equilibrium $\delta x = 6$ in. We'll use

$$\omega^2 = \frac{g}{\delta x}$$

$\delta x = 6 \text{ in} = \frac{1}{2} \text{ ft}$ in ft/sec^2, $g = 32$

$$\omega^2 = \frac{32 \ ft/sec^2}{1/2 \ ft} = 64 \ \frac{1}{sec^2}$$

So the ODE is $\quad x'' + 64x = 0.$

Let's solve for the general solution.

Characteristic eqn (use r instead of m)

$$r^2 + 64 = 0 \implies r^2 = -64 \qquad r = \pm\sqrt{-64} = \pm 8i$$

$$\alpha = 0, \quad \beta = 8$$

$$x_1 = \cos(8t), \qquad x_2 = \sin(8t)$$

$$\boxed{\begin{array}{l}\text{The general solution}\\ \\ \qquad x(t) = c_1 \cos(8t) + c_2 \sin(8t)\end{array}}$$

Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet above equilibrium with an initial downward velocity of 24 ft/sec. Find the equation of motion, the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32$ ft/sec^2.)

Let's get m and k (we'll see that again $\omega^2 = 64$)

mass: $\quad W = mg \implies 4\text{ lb} = m\left(32\ \frac{\text{ft}}{\text{sec}^2}\right)$

$$m = \frac{4\text{ lb}}{32\ \text{ft/sec}^2} = \frac{1}{8}\ \text{sec}^2\frac{\text{lb}}{\text{ft}} = \frac{1}{8}\ \text{slug}$$

get k: $\quad W = k\,\delta x \implies k = \frac{W}{\delta x} = \frac{4\text{ lb}}{\frac{1}{2}\text{ ft}} = 8\ \frac{\text{lb}}{\text{ft}}$

$$\omega^2 = \frac{k}{m} = \frac{8\ \text{lb}/\text{ft}}{\frac{1}{8}\ \text{sec}^2\ \frac{\text{lb}}{\text{ft}}} = 64\ \frac{1}{\text{sec}^2}$$

The ODE is $x'' + 64x = 0$ with general solution $x(t) = c_1 \cos(8t) + c_2 \sin(8t)$.

Given $x(0) = 4$ (4 ft above equilibrium)

$x'(0) = -24$ (24 ft/sec downward)

$$x'(t) = -8c_1 \sin(8t) + 8c_2 \cos(8t)$$

$$x(0) = c_1 \cos(0) + c_2 \sin(0) = 4 \quad \Rightarrow \quad c_1 = 4 \quad (= x_0)$$

$x'(0) = -8C_1\sin(0) + 8C_2\cos(0) = -24 \qquad 8C_2 = -24$

$C_2 = -3 \quad \left(= \frac{x_1}{\omega}\right)$

$$\boxed{\begin{array}{l}\text{The equation of motion}\\ x(t) = 4\cos(8t) - 3\sin(8t)\end{array}}$$

The period $\quad T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}$

The frequency $\quad f = \frac{1}{T} = \frac{4}{\pi}$

Lets write x as a single sine function.

$$x(t) = 4\cos(8t) - 3\sin(8t) = A\sin(8t + \phi)$$

Let $A = \sqrt{4^2 + (-3)^2} = 5$

$$x(t) = \frac{5}{5}\left(4\cos(8t) - 3\sin(8t)\right)$$

$$= 5\left(\frac{4}{5}\cos(8t) - \frac{3}{5}\sin(8t)\right)$$

Let ϕ be defined by $\sin\phi = \frac{4}{5}$, $\cos\phi = \frac{-3}{5}$

$$x(t) = 5\left(\sin\phi\cos(8t) + \cos\phi\sin(8t)\right)$$

$x(t) = 5\sin(8t + \phi)$

The amplitude $A = 5$

And phase shift $\phi \approx 2.21$ about 127°

Using the inverse cosine function

$$\phi = \cos^{-1}\left(\frac{-3}{5}\right) \approx 2.21$$

Free Damped Motion

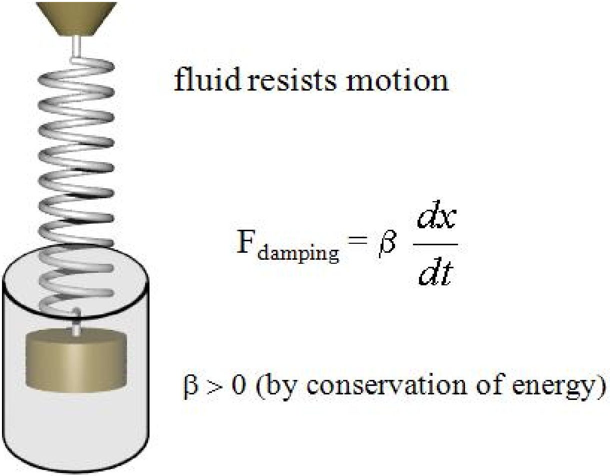

Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

Total Force = Force of spring + Force of damping

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx \quad \Longrightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Case 1: $\lambda^2 > \omega^2$ Overdamped

2 distinct real roots

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$

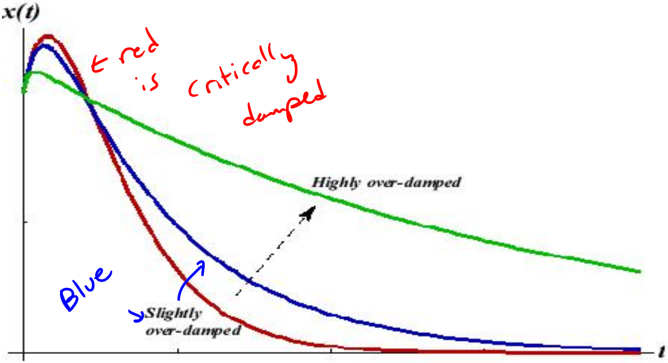

Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

Case 2: $\lambda^2 = \omega^2$ Critically Damped

One repeated real root.

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

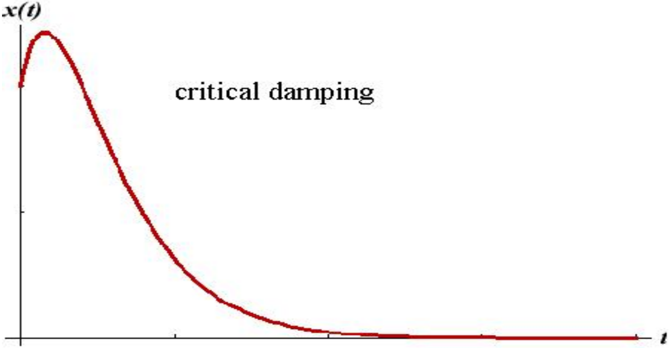

Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

Complex roots

$$x(t) = e^{-\lambda t}\left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

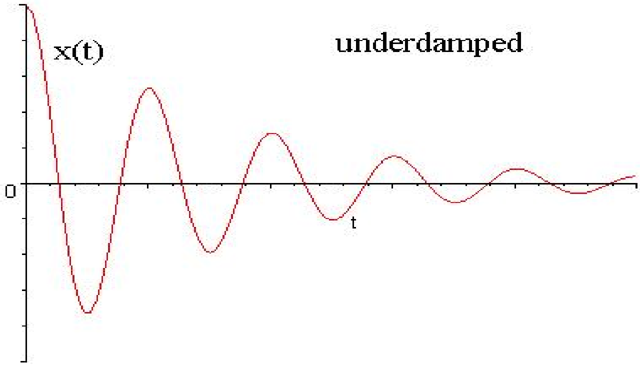

Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

Comparison of Damping

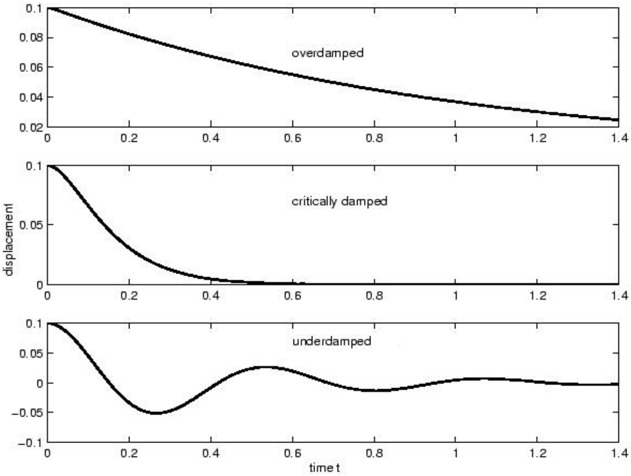

Figure: Comparison of motion for the three damping types.

Example

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

$mx'' + \beta x' + kx = 0$. we need m, β, k

Given $m = 2\,\text{kg}$, $k = 12\ \text{N/m}$

$F_{\text{damping}} = 10\frac{dx}{dt}$ so $\beta = 10$

The ODE is

$$2x'' + 10x' + 12x = 0$$

$x'' + 5x' + 6x = 0$ Charc. eqn $r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r = -2$ or $r = -3$

2 distinct real roots

The system is Over damped.

Note: $2\lambda = \frac{\beta}{m} = \frac{10}{2} = 5$, $\omega^2 = \frac{k}{m} = 6$

so $\lambda = \frac{5}{2}$

$$\lambda^2 - \omega^2 = \left(\frac{5}{2}\right)^2 - 6 = \frac{25}{4} - \frac{24}{4} = \frac{1}{4} > 0$$

$$\Rightarrow \quad \lambda^2 > \omega^2$$

the system is over damped.