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Section 4.3: Linearly Independent Sets and Bases

Definition: A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in a vector space V is said to be **linearly independent** if the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

has only the trivial solutions $c_1 = c_2 = \cdots = c_p = 0$.
The set is **linearly dependent** if there exist a nontrivial solution (at least one of the weights c_i is nonzero). If there is a nontrivial solution $c_1, \ldots, c_p$, then equation (1) is called a **linear dependence relation.**

Theorem: The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, $p \geq 2$ and $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ for $j > 1$ is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Example

Determine if the set is linearly dependent or independent in $\mathbb{P}_2$.

(a) $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ where $\mathbf{p}_1 = 1$, $\mathbf{p}_2 = 2t$, $\mathbf{p}_3 = t - 3$.

Note $\quad \vec{p}_3 = \frac{1}{2}\vec{p}_2 - 3\vec{p}_1$

so $\quad 3\vec{p}_1 - \frac{1}{2}\vec{p}_2 + \vec{p}_3 = \vec{0}$

This is a liner dependence relation with

$c_1 = 3, \quad c_2 = \frac{-1}{2}, \quad c_3 = 1.$

The set is linearly dependent.

(b) $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ where $\mathbf{p}_1 = 2,\ \mathbf{p}_2 = t,\ \mathbf{p}_3 = -t^2$.

Consider a linear combinate set to zero

$$c_1\vec{p}_1 + c_2\vec{p}_2 + c_3\vec{p}_3 = \vec{0}.$$

we have

$$2c_1 + c_2 t - c_3 t^2 = 0 + 0t + 0t^2$$

This is to hold for all real t.

When $t=0$, the equation is

$$2c_1 = 0 \implies c_1 = 0$$

When $t=1$, the equation becomes

$$c_2 - c_3 = 0 \implies c_2 = c_3$$

When $t=-1$, the equation is

$$-c_2 - c_3 = 0 \implies c_2 = -c_3$$

$$c_3 = -c_3 \implies c_3 = 0 \quad \text{so} \quad c_2 = 0 \text{ too}$$

$$c_1\vec{p}_1 + c_2\vec{p}_2 + c_3\vec{p}_3 = \vec{0}$$

has only the trivial solution $c_1 = c_2 = c_3 = 0$

The set is linearly independent in $\mathbb{P}_2$.

Example

Show that every vector $\mathbf{p} = p_0 + p_1 t + p_2 t^2$ in $\mathbb{P}_2$ can be written as a linear combination of $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$[1] where $\mathbf{p}_1 = 2$, $\mathbf{p}_2 = t$, $\mathbf{p}_3 = -t^2$.

To write $\vec{p} = p_0 + p_1 t + p_2 t^2 = c_1 \vec{p}_1 + c_2 \vec{p}_2 + c_3 \vec{p}_3$

we need $c_1 = \frac{1}{2} p_0$, $c_2 = p_1$, and $c_3 = -p_2$

$\vec{p}$ can be written as a linear combination of $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$.

[1] i.e. this set *spans* $\mathbb{P}_2$

Definition (Basis)

Definition: Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** of H provided

(i) $\mathcal{B}$ is linearly independent, and

(ii) $H =$Span$(\mathcal{B})$.

The $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ from the last example is a Basis for $\mathbb{P}_2$.

We can think of a basis as a *minimal spanning set*. All of the *information* needed to construct vectors in H is contained in the basis, and none of this information is repeated.

Example

If A is an invertible $n \times n$ matrix, then we know[2] that (1) the columns are linearly independent, and (2) the columns span $\mathbb{R}^n$. Use this to determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ where

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

We want to know if the set is ① linearly independent and ② if it spans $\mathbb{R}^3$. We can use a matrix

$A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$. Let

$$A = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix}$$

[2] from our large theorem on invertible matrices from section 2.3

Using the determinant

$$\det(A) = 3\begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix} - 0\begin{vmatrix} \cdots \end{vmatrix} + (-6)\begin{vmatrix} -4 & -2 \\ 1 & 1 \end{vmatrix}$$

$$= 3(5-7) - 6(-4+2) = -6+12 = 6$$

$-6 \neq 0$, so A is invertible. Hence $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent and it spans $\mathbb{R}^3$. Our set is a basis for $\mathbb{R}^3$.

Standard Basis in $\mathbb{R}^n$

The columns of the $n \times n$ identity matrix provide an obvious basis for $\mathbb{R}^n$. This is called the **standard basis** for $\mathbb{R}^n$. For example, the standard bases in $\mathbb{R}^2$ and $\mathbb{R}^3$ are

$$\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{e}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{e}_2} \right\}, \quad \text{and} \quad \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{e}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{e}_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{e}_3} \right\} \quad \text{respectively.}$$

Other Vector Spaces

Show that $\{1, t, t^2, t^3\}$ is a basis for $\mathbb{P}_3$[3].

First consider

$$c_1 1 + c_2 t + c_3 t^2 + c_4 t^3 = 0 + 0t + 0t^2 + 0t^3$$

requires $c_1 = c_2 = c_3 = c_4 = 0$ the trivial solution.

Hence the set is linearly independent.

Next, if $\vec{p}(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$ is any vector

in $\mathbb{P}_3$, taking $c_1 = p_0$, $c_2 = p_1$, $c_3 = p_2$ and $c_4 = p_3$

[3] The set $\{1, t, \ldots, t^n\}$ is called the **standard basis** for $\mathbb{P}_n$

then $\vec{p}(t) = c_1 \cdot 1 + c_2 t + c_3 t^2 + c_4 t^3$.

The set spans $\mathbb{P}_3$.

Hence $\{1, t, t^2, t^3\}$ is a basis for $\mathbb{P}_3$.

Other Vector Spaces

Show that $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M^{2\times 2}$.

First, if $c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow c_1 = c_2 = c_3 = c_4 = 0$ the trivial solution

The set is linearly independent.

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is any matrix in $M^{2\times 2}$, note that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence the set spans $M^{2\times 2}$.

The set is a basis for $M^{2\times 2}$.

A Spanning Set Theorem

Example: Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be vectors in a vector space V, and suppose that
(1) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and
(2) $\mathbf{v}_3 = \mathbf{v}_1 - 2\mathbf{v}_2$.
Show that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

If $\vec{v}$ is in H, then $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

for some c_1, c_2, c_3. But

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(\vec{v}_1 - 2\vec{v}_2)$$

$$= (c_1 + c_3)\vec{v}_1 + (c_2 - 2c_3)\vec{v}_2 = k_1\vec{v}_1 + k_2\vec{v}_2$$

So $\vec{v}$ is in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$.

Theorem:

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set in a vector space V and $H =$Span(S).

(a.) If one of the vectors in S, say $\mathbf{v}_k$ is a linear combination of the other vectors in S, then the subset of S obtained by eliminating $\mathbf{v}_k$ still spans H.

(b) If $H \neq \{\mathbf{0}\}$, then some subset of S is a basis for H.

If we start with a spanning set, we can eliminate *duplication* and arrive at a basis.