March 13 Math 3260 sec. 56 Spring 2018

Section 4.3: Linearly Independent Sets and Bases

Definition: A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in a vector space V is said to be **linearly independent** if the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

has only the trivial solutions $c_1 = c_2 = \cdots = c_p = 0$.
The set is **linearly dependent** if there exist a nontrivial solution (at least one of the weights c_i is nonzero). If there is a nontrivial solution $c_1, \ldots, c_p$, then equation (1) is called a **linear dependence relation.**

Theorem: The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, $p \geq 2$ and $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ for $j > 1$ is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Example

Determine if the set is linearly dependent or independent in $\mathbb{P}_2$.

(a) $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ where $\mathbf{p}_1 = 1$, $\mathbf{p}_2 = 2t$, $\mathbf{p}_3 = t - 3$.

Note that $\vec{p}_3 = \frac{1}{2}\vec{p}_2 - 3\vec{p}_1$

$\Rightarrow \quad 3\vec{p}_1 - \frac{1}{2}\vec{p}_2 + \vec{p}_3 = \vec{0}$

This is a linear dependence relation with $c_1 = 3$, $c_2 = \frac{-1}{2}$ and $c_3 = 1$ (not all zero).

The set is linearly dependent.

(b) $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ where $\mathbf{p}_1 = 2,\ \mathbf{p}_2 = t,\ \mathbf{p}_3 = -t^2$.

Consider the equation

$$c_1\vec{p}_1 + c_2\vec{p}_2 + c_3\vec{p}_3 = \vec{0}$$

$$2c_1 + c_2 t - c_3 t^2 = 0 + 0t + 0t^2$$

This must hold for all t.

When $t=0$, the equation is $2c_1 = 0 \Rightarrow c_1 = 0$

when $t=1$, the equation is $c_2 - c_3 = 0 \Rightarrow c_2 = c_3$

when $t=-1$, the equation is $-c_2 - c_3 = 0 \Rightarrow c_2 = -c_3$

$c_2 = c_3$ and $c_2 = -c_3$

$$\Rightarrow \quad c_3 = -c_3 \quad \Rightarrow \quad c_3 = 0$$

So $c_1 = 0$, $c_2 = c_3 = 0$.

The equation has only the trivial solution, hence the set is linearly independent

Example

Show that every vector $\mathbf{p} = p_0 + p_1 t + p_2 t^2$ in $\mathbb{P}_2$ can be written as a linear combination of $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$[1] where $\mathbf{p}_1 = 2,\ \mathbf{p}_2 = t,\ \mathbf{p}_3 = -t^2$.

We want to write

$$\vec{p}(t) = p_0 + p_1 t + p_2 t^2 = c_1 \vec{p}_1 + c_2 \vec{p}_2 + c_3 \vec{p}_3$$

$$p_0 + p_1 t + p_2 t^2 = 2c_1 + c_2 t - c_3 t^2$$

This holds if $c_1 = \frac{1}{2} p_0$, $c_2 = p_1$ and $c_3 = -p_2$

So each $\vec{p}$ in $\mathbb{P}_2$ is in $\mathrm{Span}\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$.

[1] i.e. this set *spans* $\mathbb{P}_2$

Definition (Basis)

Definition: Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** of H provided

(i) $\mathcal{B}$ is linearly independent, and

(ii) $H = \text{Span}(\mathcal{B})$.

$\{2, t, -t^2\}$ is a basis for $\mathbb{P}_2$ by the last two examples!

We can think of a basis as a *minimal spanning set*. All of the *information* needed to construct vectors in H is contained in the basis, and none of this information is repeated.

Example

If A is an invertible $n \times n$ matrix, then we know[2] that (1) the columns are linearly independent, and (2) the columns span $\mathbb{R}^n$. Use this to determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ where

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

We must determine if the set is linearly independent and if it spans $\mathbb{R}^3$. We can use a matrix

$A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$. Let $A = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix}$

[2] from our large theorem on invertible matrices from section 2.3

We can use the determinant.

$$\det(A) = 3\begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix} - 0\begin{vmatrix} \cdots \end{vmatrix} - 6\begin{vmatrix} -4 & -2 \\ 1 & 1 \end{vmatrix}$$

$$= 3(5-7) - 6(-4+2) = -6+12 = 6$$

$\det(A) \neq 0$, so A is invertible.

The columns (our set of vectors) is linearly independent and span $\mathbb{R}^3$.

The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

Standard Basis in $\mathbb{R}^n$

The columns of the $n \times n$ identity matrix provide an obvious basis for $\mathbb{R}^n$. This is called the **standard basis** for $\mathbb{R}^n$. For example, the standard bases in $\mathbb{R}^2$ and $\mathbb{R}^3$ are

$$\left\{ \underset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\}, \quad \text{and} \quad \left\{ \underset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{e}_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\} \quad \text{respectively.}$$

Other Vector Spaces

Show that $\{1, t, t^2, t^3\}$ is a basis for $\mathbb{P}_3$[3].

First, $c_1 1 + c_2 t + c_3 t^2 + c_4 t^3 = 0 + 0t + 0t^2 + 0t^3$

$\Rightarrow c_1 = c_2 = c_3 = c_4 = 0$ They're lin. independent

Also, for arbitrary $\vec{p}(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$ in $\mathbb{P}_3$

$\vec{p}(t) = c_1 \cdot 1 + c_2 t + c_3 t^2 + c_4 t^3$ where

$c_1 = p_0$, $c_2 = p_1$, $c_3 = p_2$, and $c_4 = p_3$

[3]The set $\{1, t, \ldots, t^n\}$ is called the **standard basis** for $\mathbb{P}_n$.

The set spans $\mathbb{P}_3$.