March 14 Math 1190 sec. 62 Spring 2017

Section 4.5: Indeterminate Forms & L'Hôpital's Rule

In this section, we are concerned with *indeterminate forms*. L'Hôpital's Rule applies directly to the forms

$$\frac{0}{0} \quad \text{and} \quad \frac{\pm\infty}{\pm\infty}.$$

Other indeterminate forms we'll encounter include

$$\infty - \infty, \quad 0 \cdot \infty, \quad 1^{\infty}, \quad 0^0, \quad \text{and} \quad \infty^0.$$

Indeterminate forms are not defined (as numbers)

Theorem: l'Hospital's Rule (part 1)

Suppose f and g are differentiable on an open interval I containing c (except possibly at c), and suppose $g'(x) \neq 0$ on I. If

$$\lim_{x\to c} f(x) = 0 \quad \text{and} \quad \lim_{x\to c} g(x) = 0$$

then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is ∞ or $-\infty$).

Use when $\lim_{x\to c} \frac{f(x)}{g(x)} = "\frac{0}{0}"$

Theorem: l'Hospital's Rule (part 2)

Suppose f and g are differentiable on an open interval I containing c (except possibly at c), and suppose $g'(x) \neq 0$ on I. If

$$\lim_{x\to c} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to c} g(x) = \pm\infty$$

then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is ∞ or $-\infty$).

Use when $\displaystyle\lim_{x\to c} \frac{f(x)}{g(x)} = \text{"}\frac{\pm\infty}{\pm\infty}\text{"}$

The form $\infty - \infty$

Evaluate the limit if possible

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \text{"}\infty - \infty\text{"}$$

$$\lim_{x\to 1^+} \frac{1}{\ln x} = \infty$$

$$\lim_{x\to 1^+} \frac{1}{x-1} = \infty$$

write $\frac{1}{\ln x} - \frac{1}{x-1}$ as one fraction

$$\frac{1}{\ln x} - \frac{1}{x-1} = \frac{x-1}{(\ln x)(x-1)} - \frac{\ln x}{(\ln x)(x-1)}$$

$$= \frac{x-1-\ln x}{(\ln x)(x-1)}$$

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1^+} \frac{x-1-\ln x}{(x-1)\ln x} = \text{"}\frac{0}{0}\text{"}$$

apply l'H rule

$$= \lim_{x\to 1^+} \frac{\frac{d}{dx}(x-1-\ln x)}{\frac{d}{dx}(x-1)\ln x}.$$

$$= \lim_{x\to 1^+} \frac{1-\frac{1}{x}}{1\cdot \ln x + (x-1)\cdot \frac{1}{x}}$$

Clear fractions

$$= \lim_{x\to 1^+} \left(\frac{1-\frac{1}{x}}{\ln x + (x-1)\cdot\frac{1}{x}}\right)\cdot\frac{x}{x}$$

$$= \lim_{x\to 1^+} \frac{x-1}{x\ln x + x - 1} = \text{"}\frac{0}{0}\text{"}$$

we can use

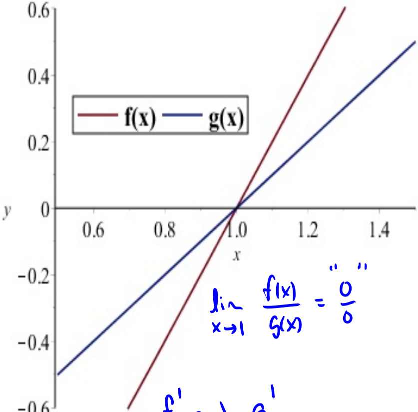
l'H rule again

$$= \lim_{x \to 1^+} \frac{\frac{d}{dx}(x-1)}{\frac{d}{dx}(x\ln x + x - 1)}$$

$$= \lim_{x \to 1^+} \frac{1}{1 \cdot \ln x + x \cdot \frac{1}{x} + 1}$$

$$= \lim_{x \to 1^+} \frac{1}{\ln x + 2} = \frac{1}{\ln 1 + 2} = \frac{1}{0+2} = \frac{1}{2}$$

Question



$\lim\limits_{x\to1} \frac{f(x)}{g(x)} = \text{"}\frac{0}{0}\text{"}$

f' and g'

tell us about slope

$y = f(x)$ and $y = g(x)$ close to $x = 1$ are plotted on the same set of axes. Note that

$$\lim_{x\to1} f(x) = 0 \quad \text{and} \quad \lim_{x\to1} g(x) = 0$$

From the graph, only one of the following limit statements could be true. Which one?

(a) $\lim\limits_{x\to1} \frac{f(x)}{g(x)} = 0$

(b) $\lim\limits_{x\to1} \frac{f(x)}{g(x)} = 2$

(c) $\lim\limits_{x\to1} \frac{f(x)}{g(x)} = -2$

Slopes of f and g are both positive.

I'Hospital's Rule is not a "Fix-all"

Evaluate $\lim_{x\to 0^+} \frac{\cot x}{\csc x}$ $= \text{"}\frac{\infty}{\infty}\text{"}$

Recall

$\lim_{x\to 0^+} \cot x = \infty$

and

$\lim_{x\to 0^+} \csc x = \infty$

apply l'H rule

$$\lim_{x\to 0^+} \frac{\cot x}{\csc x} = \lim_{x\to 0^+} \frac{\frac{d}{dx}\cot x}{\frac{d}{dx}\csc x}$$

$$= \lim_{x\to 0^+} \frac{-\csc^2 x}{-\csc x \cot x}$$

$$= \lim_{x\to 0^+} \frac{\csc x}{\cot x} = \text{"}\frac{\infty}{\infty}\text{"}$$

Let use l'H rule again

$$= \lim_{x\to 0^+} \frac{\frac{d}{dx}\csc x}{\frac{d}{dx}\cot x} = \lim_{x\to 0^+} \frac{-\csc x \cot x}{-\csc^2 x}$$

$$= \lim_{x\to 0^+} \frac{\cot x}{\csc x}$$

l'H rule doesn't help here.

Let's use trig IDs

$$\lim_{x\to 0^+} \frac{\cot x}{\csc x} = \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}} = \lim_{x\to 0^+} \frac{\cos x}{\cancel{\sin x}} \cdot \frac{\cancel{\sin x}}{1}$$

$$= \lim_{x\to 0^+} \cos x = \cos 0 = 1$$

Don't apply it if it doesn't apply!

$$\lim_{x\to 2} \frac{x+4}{x^2-3} = \frac{6}{1} = 6$$

BUT

$$\lim_{x\to 2} \frac{\frac{d}{dx}(x+4)}{\frac{d}{dx}(x^2-3)} \quad = \quad \lim_{x\to 2} \frac{1}{2x} \quad = \quad \frac{1}{4}$$

Remarks:

- l'Hopital's rule **only applies directly** to the forms $0/0$, or $(\pm\infty)/(\pm\infty)$.
- Multiple applications may be needed, or it may not result in a solution.
- It can be applied indirectly to the form $0 \cdot \infty$ or $\infty - \infty$ by rewriting the expression as a quotient.
- Derivatives of numerator and denominator are taken **separately**–this is NOT a *quotient rule* application.
- Applying it where it doesn't belong likely produces nonsense!

Question

True or False: If $\lim_{x \to c} f(x)g(x)$ produces the indeterminate form

$$0 \cdot \infty$$

then we apply l'Hopital's rule by considering

$$\lim_{x \to c} f'(x) \cdot g'(x)$$

The rule is for "$\frac{0}{0}$" or "$\frac{\pm\infty}{\pm\infty}$"

Indeterminate Forms 1^∞, 0^0, and ∞^0

Since the logarithm and exponential functions are continuous, and $\ln(x^r) = r \ln x$, we have

$$\lim_{x\to a} F(x) = \exp\left(\ln\left[\lim_{x\to a} F(x)\right]\right) = \exp\left(\lim_{x\to a} \ln F(x)\right)$$

provided this limit exists.

- we want $\lim_{x\to a} F(x)$

- instead, we compute $\lim_{x\to a} \ln(F(x)) = L$

- then $\lim_{x\to a} F(x) = e^L$

Use this property to show that

$$\lim_{x\to 0}(1+x)^{1/x} = e$$

Note $\lim_{x\to 0}(1+x) = 1$

$\lim_{x\to 0} \frac{1}{x}$: an infinity

$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \text{"}1^{\infty}\text{"}$

Here $F(x) = (1+x)^{\frac{1}{x}}$. Take the log : $\ln F(x) = \ln(1+x)^{\frac{1}{x}}$

$= \frac{1}{x}\ln(1+x)$

Take $\lim_{x\to 0} \ln(F(x))$

$$\lim_{x\to 0} \ln(1+x)^{\frac{1}{x}} = \lim_{x\to 0} \frac{1}{x}\ln(1+x)$$

$$= \lim_{x\to 0} \frac{\ln(1+x)}{x} = \text{"} \frac{0}{0} \text{"} \quad \text{use l'H rule}$$

$$= \lim_{x\to 0} \frac{\frac{d}{dx}\ln(1+x)}{\frac{d}{dx}\, x}$$

$$= \lim_{x\to 0} \frac{\frac{1}{x+1}}{1} = \frac{\frac{1}{0+1}}{1} = \frac{\frac{1}{1}}{1} = 1$$

we have $\lim_{x\to 0} \ln(1+x)^{\frac{1}{x}} = 1$

to get our limit, we exponentiate.

So $$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e^{1} = e$$

Question

True or False: Since $1^n = 1$ for every integer n, we should conclude that the indeterminate form 1^∞ is equal to 1.

No, we just saw an example where the limit was e, and $e \neq 1$.

Question

The limit

$$\lim_{x\to\infty} x^{1/x} \quad \text{gives rise to the indeterminate form}$$

(a) $\frac{\infty}{\infty}$

(b) ∞^0

(c) 0^0

(d) 1^∞

$$\left.\begin{array}{l}\lim_{x\to\infty} x = \infty \\ \lim_{x\to\infty} \frac{1}{x} = 0\end{array}\right\} \Rightarrow \lim_{x\to\infty} x^{\frac{1}{x}} = \text{"}\infty^0\text{"}$$

we have a power indet. form.

we'll try taking the limit of

$$\ln x^{\frac{1}{x}}$$

Question

$$\text{Since} \quad \ln\left(x^{1/x}\right) = \frac{1}{x}\ln x = \frac{\ln x}{x}, \quad \text{evaluate} \quad \lim_{x\to\infty}\frac{\ln x}{x}$$

(use l'Hopital's rule as needed)

$$\lim_{x\to\infty}\frac{\ln x}{x} = \text{"}\frac{\infty}{\infty}\text{"} \quad \text{use l'H rule}$$

$$= \lim_{x\to\infty}\frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \frac{0}{1} = 0$$

(a) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x} = 0$

(b) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x} = 1$

(c) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x} = \infty$

Question

$$\lim_{x\to\infty} x^{1/x} =$$

we found $\lim_{x\to\infty} \ln x^{\frac{1}{x}} = 0$

so

$$\lim_{x\to\infty} x^{\frac{1}{x}} = e^{0} = 1$$

(a) 0

(b) 1

(c) ∞

Section 4.2: Maximum and Minimum Values; Critical Numbers

Definition: Let f be a function with domain D and let c be a number in D. Then $f(c)$ is

- **the absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D,
- **the absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D.

Note that if an absolute minimum occurs at c, then $f(c)$ is the **absolute minimum value** of f. Similarly, if an absolute maximum occurs at c, then $f(c)$ is the **absolute maximum value** of f.

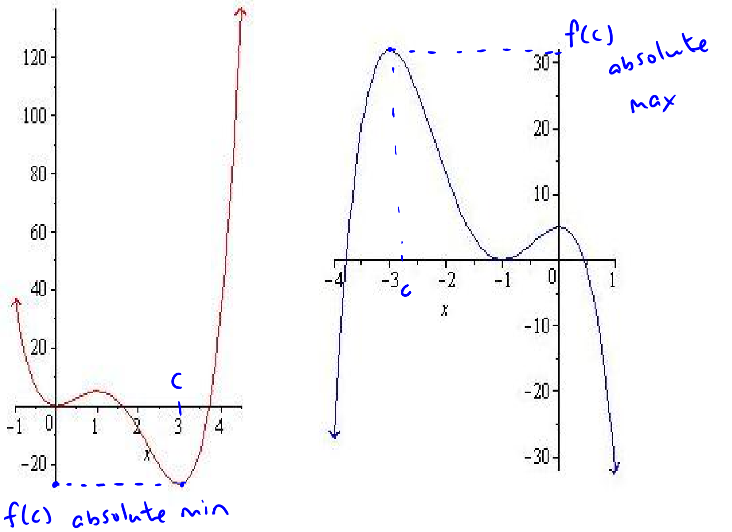


Figure: Graphically, an absolute minimum is the lowest point and an absolute maximum is the highest point.

Local Maximum and Minimum

Definition: Let f be a function with domain D and let c be a number in D. Then $f(c)$ is

- **a local minimum** value of f if $f(c) \leq f(x)$ for x *near** c
- **a local maximum** value of f if $f(c) \geq f(x)$ for x *near* c.

More precisely, to say that x is *near* c means that there exists an open interval containing c such that for all x in this interval the respective inequality holds.

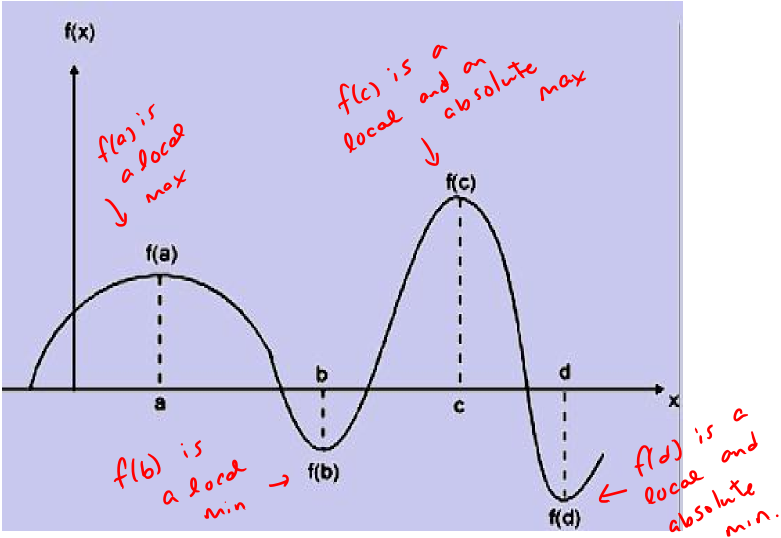


Figure: Graphically, local maxes and mins are *relative* high and low points.

Terminology

Maxima—-plural of maximum

Minima—-plural of minimum

Extremum——is either a maximum or a minimum

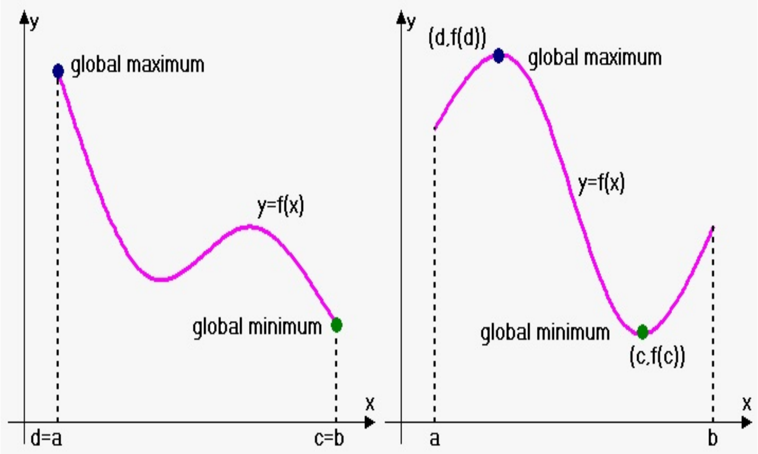
Extrema——plural of extremum

”**Global**” is another word for absolute.

”**Relative**” is another word for local.

Extreme Value Theorem (EVT)

Suppose f is continuous on a closed interval $[a, b]$. Then f attains an absolute maximum value $f(d)$ and f attains an absolute minimum value $f(c)$ for some numbers c and d in $[a, b]$.



Fermat's Theorem

Note that the Extreme Value Theorem tells us that a continuous function is guaranteed to take an absolute maximum and absolute minimum on a closed interval. It does not provide a method for actually finding these values or where they occur. For that, the following theorem due to Fermat is helpful.

Theorem: If f has a local extremum at c and if $f'(c)$ exists, then

$$f'(c) = 0.$$

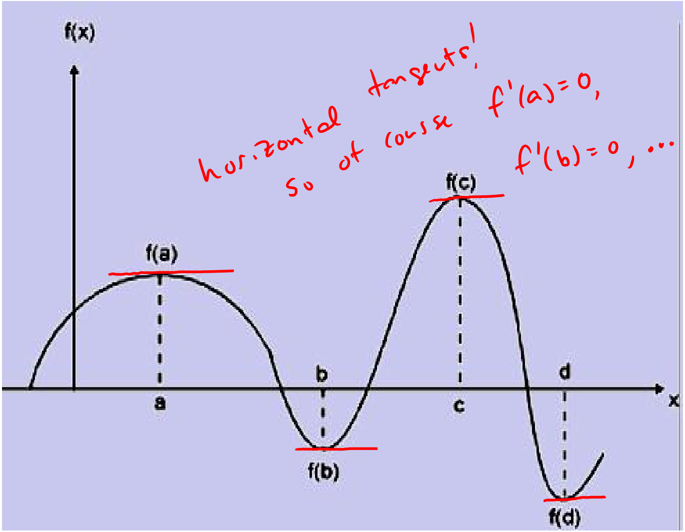


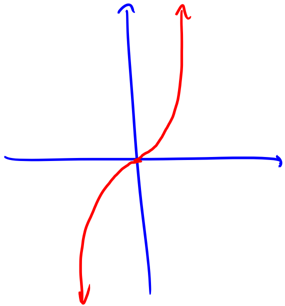
Figure: We note that at the local extrema, the tangent line would be horizontal.

Is the Converse of our Theorem True?

Suppose a function f satisfies $f'(0) = 0$. Can we conclude that $f(0)$ is a local maximum or local minimum?

Turns out, the answer is no.

Consider $f(x) = x^3$. $f'(x) = 3x^2$ and $f'(0) = 3 \cdot 0^2 = 0$



Does an extremum have to correspond to a horizontal tangent?

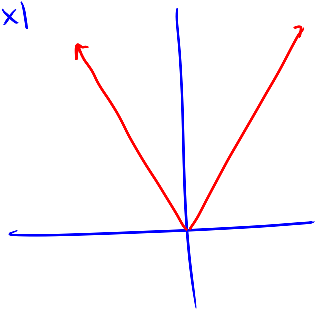
Could $f(c)$ be a local extremum but have $f'(c)$ not exist?

Yes, $f'(c)$ need not exist.

Take for example $f(x) = |x|$

Has a local (and global) min @ $(0,0)$, but

$f'(0)$ DNE



Critical Number

Definition: A **critical number** of a function f is a number c in its domain such that either

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

Theorem: If f has a local extremum at c, then c is a critical number of f.

Some authors call critical numbers *critical points*.

Example

Find all of the critical numbers of the function.

$$g(t) = t^{1/5}(12 - t)$$

The domain is $(-\infty, \infty)$

we need to find where $g'(t) = 0$ or $g'(t)$ DNE

$g(t) = 12\,t^{1/5} - t^{6/5}$

find g'

$g'(t) = \frac{12}{5}\,t^{-4/5} - \frac{6}{5}\,t^{1/5}$

factor

$g'(t) = \frac{6}{5}\,t^{-4/5}\,(2 - t) = \dfrac{6(2-t)}{5\,t^{4/5}}$

$g'(t) = 0$ if the numerator is zero

$$6(2-t) = 0 \implies t = 2$$

$g'(t)$ DNE if the denominator is zero

$$5t^{4/5} = 0 \implies t = 0$$

Our critical numbers are 2 and 0.