March 14 Math 1190 sec. 63 Spring 2017

Section 4.5: Indeterminate Forms & L'Hôpital's Rule

In this section, we are concerned with *indeterminate forms*. L'Hôpital's Rule applies directly to the forms

$$\frac{0}{0} \quad \text{and} \quad \frac{\pm\infty}{\pm\infty}.$$

Other indeterminate forms we'll encounter include

$$\infty-\infty, \quad 0\cdot\infty, \quad 1^{\infty}, \quad 0^0, \quad \text{and} \quad \infty^0.$$

Indeterminate forms are not defined (as numbers)

Theorem: l'Hospital's Rule (part 1)

Suppose f and g are differentiable on an open interval I containing c (except possibly at c), and suppose $g'(x) \neq 0$ on I. If

$$\lim_{x\to c} f(x) = 0 \quad \text{and} \quad \lim_{x\to c} g(x) = 0$$

then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is ∞ or $-\infty$).

Use when $\displaystyle\lim_{x\to c} \frac{f(x)}{g(x)} = \text{"}\frac{0}{0}\text{"}$

Theorem: l'Hospital's Rule (part 2)

Suppose f and g are differentiable on an open interval I containing c (except possibly at c), and suppose $g'(x) \neq 0$ on I. If

$$\lim_{x\to c} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to c} g(x) = \pm\infty$$

then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is ∞ or $-\infty$).

Use when $\displaystyle\lim_{x\to c} \frac{f(x)}{g(x)} = \text{"} \frac{\pm\infty}{\pm\infty} \text{"}$

The form $\infty - \infty$

Evaluate the limit if possible

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) = \text{"}\infty - \infty\text{"}$$

Note $\displaystyle\lim_{x\to 1^+} \frac{1}{\ln x} = \infty$

$\displaystyle\lim_{x\to 1^+} \frac{1}{x-1} = \infty$

write $\frac{1}{\ln x} - \frac{1}{x-1}$ as one fraction

$$\frac{1}{\ln x} - \frac{1}{x-1} = \frac{x-1}{(x-1)\ln x} - \frac{\ln x}{(x-1)\ln x} = \frac{x-1-\ln x}{(x-1)\ln x}$$

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1^+}\frac{x-1-\ln x}{(x-1)\ln x} = \text{"}\frac{0}{0}\text{"}$$

apply l'H rule

$$\lim_{x\to 1^+} \frac{x-1-\ln x}{(x-1)\ln x} = \lim_{x\to 1^+} \frac{\frac{d}{dx}(x-1-\ln x)}{\frac{d}{dx}(x-1)\ln x}$$

$$= \lim_{x\to 1^+} \frac{1-0-\frac{1}{x}}{1\cdot\ln x + (x-1)\cdot\frac{1}{x}}$$

Clear the fraction

$$= \lim_{x\to 1^+} \left(\frac{1-\frac{1}{x}}{\ln x + (x-1)\frac{1}{x}}\right)\frac{x}{x}$$

$$= \lim_{x\to 1^+} \frac{x-1}{x\ln x + (x-1)} = \text{"}\frac{0}{0}\text{"}$$

apply l'H again

$$= \lim_{x\to 1^+} \frac{\frac{d}{dx}(x-1)}{\frac{d}{dx}(x\ln x + x - 1)}$$

$$= \lim_{x\to 1^+} \frac{1}{1\cdot \ln x + x\cdot\frac{1}{x} + 1}$$

$$= \lim_{x\to 1^+} \frac{1}{\ln x + 2} = \frac{1}{\ln 1 + 2} = \frac{1}{0+2} = \frac{1}{2}$$

Recall

$$\frac{d}{dx}a^x = a^x \ln(a)$$

and

$$a^0 = 1$$

for every $a > 0$ with $a \neq 1$.

Question

Use L'Hôpital's Rule to evaluate the limit

$$\lim_{x\to 0} \frac{2^x - 3^x}{x} \quad = \text{"}\frac{0}{0}\text{"}$$

use l'H rule

$$\lim_{x\to 0} \frac{\frac{d}{dx}(2^x - 3^x)}{\frac{d}{dx}x}$$

$$= \lim_{x\to 0} \frac{2^x \ln 2 - 3^x \ln 3}{1}$$

$$= 2^0 \ln 2 - 3^0 \ln 3 = \ln 2 - \ln 3$$

(a) $\ln(-1)$

(b) -1

(c) $\ln(2) - \ln(3)$

(d) $-\infty$

l'Hospital's Rule is not a "Fix-all"

Evaluate $\lim_{x\to 0^+} \frac{\cot x}{\csc x}$ $=$ "$\frac{\infty}{\infty}$"

$\lim_{x\to 0^+} \cot x = \infty$

$\lim_{x\to 0^+} \csc x = \infty$

Apply l'H rule

$$\lim_{x\to 0^+} \frac{\cot x}{\csc x} = \lim_{x\to 0^+} \frac{\frac{d}{dx}\cot x}{\frac{d}{dx}\csc x}$$

$$= \lim_{x\to 0^+} \frac{-\csc^2 x}{-\csc x \cot x}$$

$$= \lim_{x\to 0^+} \frac{\csc x}{\cot x} = \text{"}\frac{\infty}{\infty}\text{"}$$

use l'H again

$$= \lim_{x\to 0^+} \frac{\frac{d}{dx}\csc x}{\frac{d}{dx}\cot x} = \lim_{x\to 0^+} \frac{-\csc x\cot x}{-\csc^2 x}$$

$$= \lim_{x\to 0^+} \frac{\cot x}{\csc x}$$ l'H rule doesn't help here

Use trig ID's

$$\lim_{x\to 0^+} \frac{\cot x}{\csc x} = \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}} = \lim_{x\to 0^+} \frac{\cos x}{\cancel{\sin x}} \cdot \frac{\cancel{\sin x}}{1}$$

$$= \lim_{x\to 0^+} \cos x = \cos 0 = 1$$

Don't apply it if it doesn't apply!

$$\lim_{x\to 2} \frac{x+4}{x^2-3} = \frac{6}{1} = 6$$

BUT

$$\lim_{x\to 2} \frac{\frac{d}{dx}(x+4)}{\frac{d}{dx}(x^2-3)} \quad = \quad \lim_{x\to 2} \frac{1}{2x} \quad = \quad \frac{1}{4}$$

Remarks:

- l'Hopital's rule **only applies directly** to the forms $0/0$, or $(\pm\infty)/(\pm\infty)$.
- Multiple applications may be needed, or it may not result in a solution.
- It can be applied indirectly to the form $0 \cdot \infty$ or $\infty - \infty$ by rewriting the expression as a quotient.
- Derivatives of numerator and denominator are taken **separately**–this is NOT a *quotient rule* application.
- Applying it where it doesn't belong likely produces nonsense!

Question

True or False: If $\lim_{x \to c} f(x)g(x)$ produces the indeterminate form

$$0 \cdot \infty$$

then we apply l'Hopital's rule by considering

$$\lim_{x \to c} f'(x) \cdot g'(x)$$

No, it applies to the forms $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

Indeterminate Forms 1^∞, 0^0, and ∞^0

Since the logarithm and exponential functions are continuous, and $\ln(x^r) = r\ln x$, we have

$$\lim_{x\to a} F(x) = \exp\left(\ln\left[\lim_{x\to a} F(x)\right]\right) = \exp\left(\lim_{x\to a} \ln F(x)\right)$$

provided this limit exists.

- we want is $\lim_{x\to a} F(x)$

- instead, we find $\lim_{x\to a} \ln(F(x)) = L$

- then exponentiate $\lim_{x\to a} F(x) = e^L$

Use this property to show that

$$\lim_{x\to 0}(1+x)^{1/x} = e$$

Note $\lim_{x\to 0}(1+x) = 1$

$\lim_{x\to 0^-} \frac{1}{x} = -\infty$ $\lim_{x\to 0^+} \frac{1}{x} = \infty$

$\lim_{x\to 0}(1+x)^{\frac{1}{x}} =$ " 1^∞ "

Here $F(x) = (1+x)^{\frac{1}{x}}$. Take the log

$$\ln F(x) = \ln (1+x)^{\frac{1}{x}} = \frac{1}{x}\ln(1+x) = \frac{\ln(1+x)}{x}$$

Now take $\lim_{x\to 0} \ln F(x)$

$$\lim_{x\to 0} \ln(1+x)^{\frac{1}{x}} = \lim_{x\to 0} \frac{\ln(1+x)}{x} = \text{“}\frac{0}{0}\text{”} \quad \text{use l'H rule}$$

$$= \lim_{x\to 0} \frac{\frac{d}{dx}\ln(1+x)}{\frac{d}{dx}x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = \frac{\frac{1}{1+0}}{1} = \frac{\frac{1}{1}}{1} = 1$$

We found that $\lim_{x\to 0} \ln(1+x)^{\frac{1}{x}} = 1$

we wanted $\lim_{x\to 0} (1+x)^{\frac{1}{x}}$

Exponentiate $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e^{1} = e$

Question

True or False: Since $1^n = 1$ for every integer n, we should conclude that the indeterminate form 1^∞ is equal to 1.

No, if it were 1, it wouldn't be indeterminate.

In our last example, the form 1^∞ corresponded to a limit of $e \neq 1$.

Question

The limit

$$\lim_{x\to\infty} x^{1/x} \quad \text{gives rise to the indeterminate form}$$

(a) $\frac{\infty}{\infty}$

(b) ∞^0

(c) 0^0

(d) 1^∞

$$\left.\begin{aligned}\lim_{x\to\infty} x &= \infty \\ \lim_{x\to\infty} \frac{1}{x} &= 0\end{aligned}\right\} \Rightarrow \lim_{x\to\infty} x^{\frac{1}{x}} = \text{``}\infty^0\text{''}$$

To evaluate this limit, we'll consider the limit of $\ln\left(x^{\frac{1}{x}}\right)$

Question

$$\text{Since} \quad \ln\left(x^{1/x}\right) = \frac{1}{x}\ln x = \frac{\ln x}{x}, \quad \text{evaluate} \quad \lim_{x\to\infty} \frac{\ln x}{x}$$

(use l'Hopital's rule as needed)

(a) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x} = 0$

(b) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x} = 1$

(c) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x} = \infty$

$$\lim_{x\to\infty} \frac{\ln x}{x} = \text{"}\frac{\infty}{\infty}\text{"} \quad \text{use l'H rule}$$

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \frac{0}{1} = 0$$

Question

$$\lim_{x\to\infty} x^{1/x} =$$

(a) 0

(b) 1

(c) ∞

we found

$$\lim_{x\to\infty} \ln\left(x^{\frac{1}{x}}\right) = 0$$

so

$$\lim_{x\to\infty} x^{\frac{1}{x}} = e^0 = 1$$

Section 4.2: Maximum and Minimum Values; Critical Numbers

Definition: Let f be a function with domain D and let c be a number in D. Then $f(c)$ is

- **the absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D,

- **the absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D.

Note that if an absolute minimum occurs at c, then $f(c)$ is the **absolute minimum value** of f. Similarly, if an absolute maximum occurs at c, then $f(c)$ is the **absolute maximum value** of f.

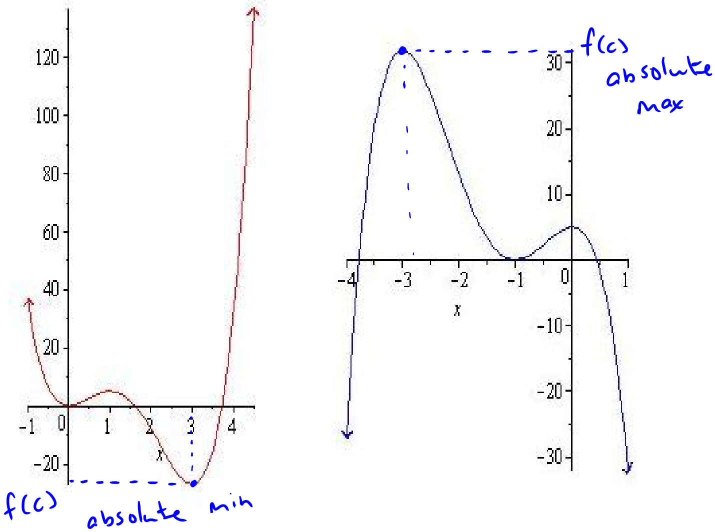


Figure: Graphically, an absolute minimum is the lowest point and an absolute maximum is the highest point.

Local Maximum and Minimum

Definition: Let f be a function with domain D and let c be a number in D. Then $f(c)$ is

- **a local minimum** value of f if $f(c) \leq f(x)$ for x *near** c
- **a local maximum** value of f if $f(c) \geq f(x)$ for x *near* c.

More precisely, to say that x is *near* c means that there exists an open interval containing c such that for all x in this interval the respective inequality holds.

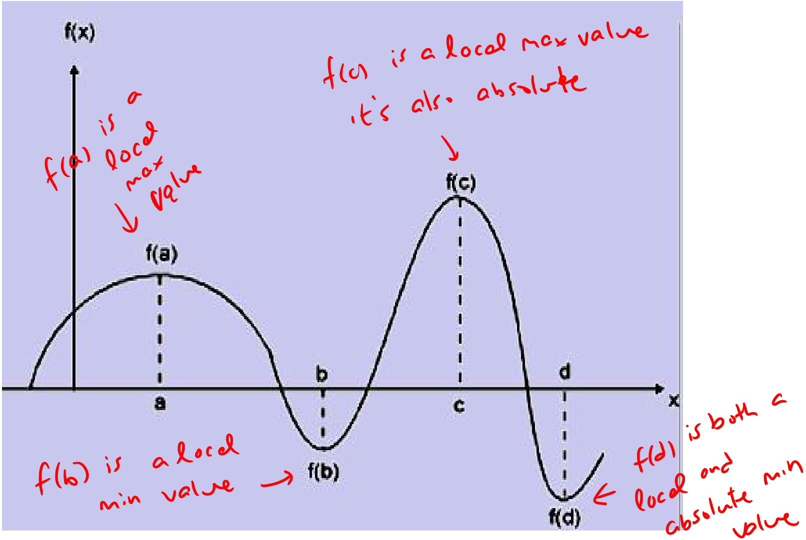


Figure: Graphically, local maxes and mins are *relative* high and low points.

Terminology

Maxima—-plural of maximum

Minima—-plural of minimum

Extremum——is either a maximum or a minimum

Extrema——plural of extremum

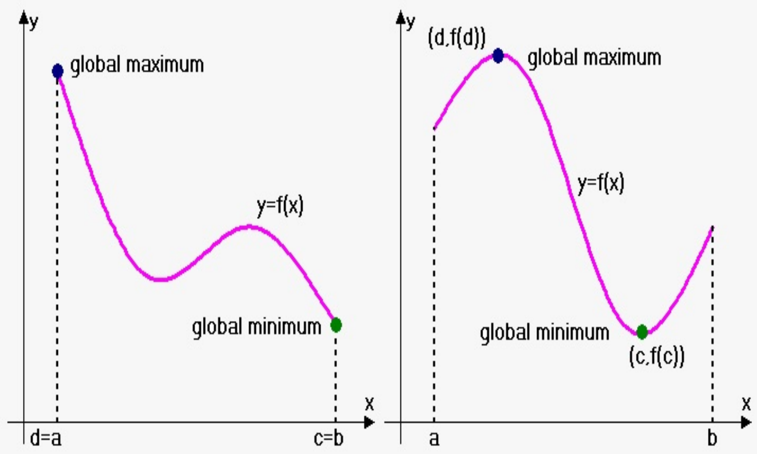
”**Global**” is another word for absolute.

”**Relative**” is another word for local.

Extreme Value Theorem

EVT

Suppose f is continuous on a closed interval $[a, b]$. Then f attains an absolute maximum value $f(d)$ and f attains an absolute minimum value $f(c)$ for some numbers c and d in $[a, b]$.



Fermat's Theorem

Note that the Extreme Value Theorem tells us that a continuous function is guaranteed to take an absolute maximum and absolute minimum on a closed interval. It does not provide a method for actually finding these values or where they occur. For that, the following theorem due to Fermat is helpful.

Theorem: If f has a local extremum at c and if $f'(c)$ exists, then

$$f'(c) = 0.$$

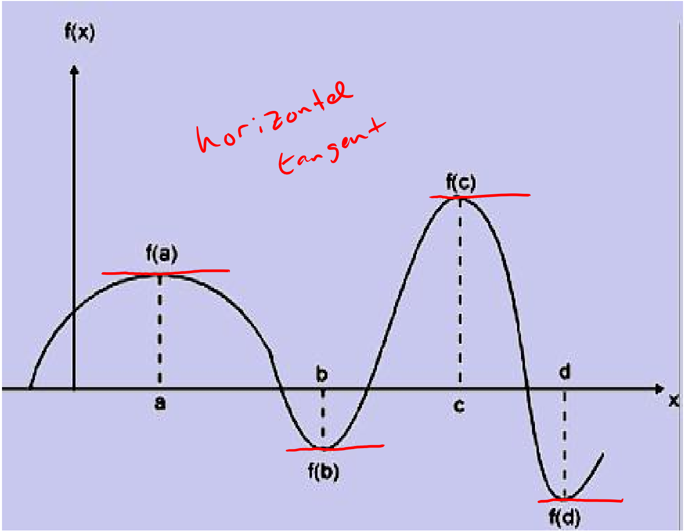


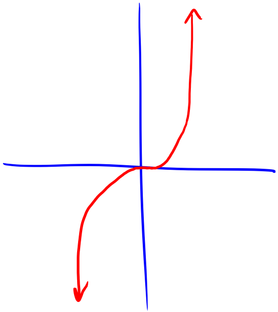
Figure: We note that at the local extrema, the tangent line would be horizontal.

Is the Converse of our Theorem True?

Suppose a function f satisfies $f'(0) = 0$. Can we conclude that $f(0)$ is a local maximum or local minimum?

No, it need not be an extreme value.

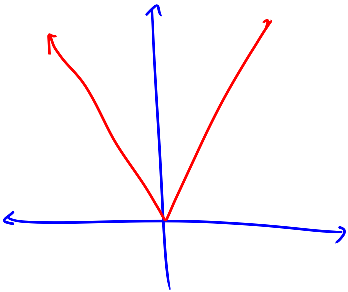
Consider $f(x) = x^3$, $f'(x) = 3x^2$ and $f'(0) = 3 \cdot 0^2 = 0$



Does an extremum have to correspond to a horizontal tangent?

Could $f(c)$ be a local extremum but have $f'(c)$ not exist?

Yes, $f'(c)$ need not exist. Consider $f(x) = |x|$



f takes a local (and global) minimum value of $f(0) = 0$.

But $f'(0)$ DNE

Critical Number

Definition: A **critical number** of a function f is a number c in its domain such that either

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

Theorem: If f has a local extremum at c, then c is a critical number of f.

Some authors call critical numbers *critical points*.

Example

Find all of the critical numbers of the function.

$$g(t) = t^{1/5}(12 - t)$$

The domain of g is $(-\infty, \infty)$.

We need to know for which t values

$g'(t) = 0$ or $g'(t)$ DNE

Find $g'(t)$: $g(t) = 12t^{1/5} - t^{6/5}$

$g'(t) = \frac{12}{5}t^{-4/5} - \frac{6}{5}t^{1/5}$ Let's factor

$$g'(t) = \frac{6}{5} t^{-4/5} \left(2 - t \right) = \frac{6(2-t)}{5\, t^{4/5}}$$

$g'(t) = 0$ if the numerator is zero.

$$6(2-t) = 0 \implies t = 2.$$

$g'(t)$ DNE if the denominator is zero.

$$5t^{4/5} = 0 \implies t = 0$$

g has two critical numbers, 0 and 2.