March 15 Math 2306 sec 58 Spring 2016

Section 9: Method of Undetermined Coefficients

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply that term by x^n, where n is the smallest positive integer that eliminate the duplication.

Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Find y_c: $\quad y'' - 4y' + 4y = 0$

Ch. eqn $\quad m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \quad m = 2$ repeated

$y_1 = e^{2x}, \quad y_2 = xe^{2x}$

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

For y_p consider $\quad y'' - 4y' + 4y = \sin(4x) \quad g_1(x)$

and $\quad y'' - 4y' + 4y = xe^{2x} \quad g_2(x)$

For $g_1(x) = \sin 4x$ guess

$$y_{p_1} = A\sin(4x) + B\cos(4x)$$

This will work as written.

For $g_2(x) = xe^{2x}$

$$y_{p_2} = (Cx + D)e^{2x} = Cxe^{2x} + De^{2x}$$

won't work, these duplicate y_c

Update $y_{p_2} = (Cx+D)xe^{2x} = Cx^2e^{2x} + Dxe^{2x}$

Still won't work due to Dxe^{2x} term

Update again $y_{p_2} = (Cx+D)x^2e^{2x} = Cx^3e^{2x} + Dx^2e^{2x}$

This works

So $y_p = A\sin(4x) + B\cos(4x) + Cx^3e^{2x} + Dx^2e^{2x}$

Find the form of the particular soluition

$$y''' - y'' + y' - y = \cos x + x^4$$

Find y_c: $\quad y''' - y'' + y' - y = 0$

Ch. eqn $\quad m^3 - m^2 + m - 1 = 0$

factor by grouping $\quad m^2(m-1) + (m-1) = 0$

$$(m-1)(m^2+1) = 0$$

$\Rightarrow \; m = 1 \quad$ or $\quad m^2 + 1 = 0 \;\Rightarrow\; m = \pm i$

$\alpha \pm i\beta$

$\alpha = 0 \quad \beta = 1$

$$y_1 = e^{x}, \quad y_2 = e^{0x}\cos x = \cos x, \quad y_3 = e^{0x}\sin x = \sin x$$

$$y_c = C_1 e^{x} + C_2 \cos x + C_3 \sin x$$

Let $g_1(x) = \cos x$

$$y_{p_1} = A\cos x + B\sin x$$

won't work, duplicates y_2, y_3 in y_c

Try again $\quad y_{p_1} = (A\cos x + B\sin x)x = Ax\cos x + Bx\sin x$

This will work.

Let $g_2(x) = x^4$

$$y_{p_2} = Cx^4 + Dx^3 + Ex^2 + Fx + G$$

this works

$$y_p = Ax\cos x + Bx\sin x + Cx^4 + Dx^3 + Ex^2 + Fx + G$$

Find the form of the particular soluition

$$y'' - 2y' + 5y = e^x + 7\sin(2x)$$

Find y_c: $\quad y'' - 2y' + 5y = 0$

Ch. eqn $\quad m^2 - 2m + 5 = 0$

Complete the square $\quad m^2 - 2m + 1 - 1 + 5 = 0$

$$(m-1)^2 + 4 = 0$$

$$(m-1)^2 = -4 \Rightarrow m - 1 = \pm 2i$$

$m = 1 \pm 2i \qquad \alpha = 1, \; \beta = 2$

$\alpha \pm i\beta$

$$y_1 = e^x \cos(2x), \quad y_2 = e^x \sin(2x)$$

$$y_c = c_1 e^x \cos(2x) + c_2 e^x \sin(2x)$$

Let $g_1(x) = e^x$ $\qquad y_{p_1} = Ae^x$ $\qquad$ This works as written.

Let $g_2(x) = 7\sin(2x)$ $\qquad y_{p_2} = B\sin(2x) + C\cos(2x)$

works as written.

$$y_p = Ae^x + B\sin(2x) + C\cos(2x).$$

Solve the IVP

$$y'' - 4y' + 4y = 8x - 4 \quad y(0) = 3, \quad y'(0) = -2$$

From before $\quad y_c = c_1 e^{2x} + c_2 x e^{2x}$

Find y_p: From $g(x) = 8x - 4$, guess $y_p = Ax + B$

$y_p = Ax + B$

$y_p' = A$

$y_p'' = 0$

$y_p'' - 4y_p' + 4y_p = 8x - 4$

$0 - 4(A) + 4(Ax + B) = 8x - 4$

$4Ax + (-4A + 4B) = 8x - 4$

$$4A = 8 \Rightarrow A = 2$$

$$-4A + 4B = -4 \Rightarrow 4B = -4 + 4A = -4 + 8 = 4 \Rightarrow B = 1$$

so $y_p = 2x + 1$

The general solution is

$$y = C_1 e^{2x} + C_2 x e^{2x} + 2x + 1$$

Apply $y(0) = 3$, $y'(0) = -2$

$$y' = 2C_1 e^{2x} + C_2 e^{2x} + 2C_2 x e^{2x} + 2$$

$$y(0) = c_1 e^0 + c_2 \cdot 0 e^0 + 2\cdot 0 + 1 = 3$$

$$c_1 + 1 = 3 \Rightarrow c_1 = 2$$

$$y'(0) = 2c_1 e^0 + c_2 e^0 + 2c_2 \cdot 0 e^0 + 2 = -2$$

$$2c_1 + c_2 + 2 = -2$$

$$c_2 = -2 - 2 - 2c_1 = -4 - 4 = -8$$

The soln to the IVP is

$$y = 2e^{2x} - 8xe^{2x} + 2x + 1.$$

Solve the IVP

$$y'' - y = 4e^{-x} \quad y(0) = -1, \quad y'(0) = 1$$

Find y_c: Ch. eqn $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$y_1 = e^{x}$, $y_2 = e^{-x}$ $\quad y_c = C_1 e^{x} + C_2 e^{-x}$

Find y_p: $y_p = Ae^{-x}$ won't work

Try $y_p = (Ae^{-x})x = Axe^{-x}$ This will work.

$$y_p = Axe^{-x}$$

$$y_p' = Ae^{-x} - Axe^{-x}$$

$$y_p'' = -Ae^{-x} - Ae^{-x} + Axe^{-x} = -2Ae^{-x} + Axe^{-x}$$

$$y_p'' - y_p = 4e^{-x}$$

$$-2Ae^{-x} + Axe^{-x} - Axe^{-x} = 4e^{-x}$$

$$-2Ae^{-x} = 4e^{-x} \Rightarrow A = -2$$

$$y_p = -2xe^{-x}$$

General Solution

$$y = c_1 e^{x} + c_2 e^{-x} - 2x e^{-x}$$

Apply $y(0) = -1$, $y'(0) = 1$

$$y' = c_1 e^{x} - c_2 e^{-x} - 2e^{-x} + 2x e^{-x}$$

$$y(0) = c_1 e^{0} + c_2 e^{0} - 2\cdot 0 e^{0} = -1 \Rightarrow \quad c_1 + c_2 = -1$$

$$y'(0) = c_1 e^{0} - c_2 e^{0} - 2e^{0} + 2\cdot 0 e^{0} = 1 \Rightarrow \quad c_1 - c_2 = 3$$

add $\quad 2c_1 = 2$

$$c_1 = 1$$

$$c_2 = -1 - c_1 = -1 - 1 = -2$$

$$\boxed{\text{The soln to the IVP is} \quad y = e^{x} - 2e^{-x} - 2xe^{-x}.}$$

Section 10: Variation of Parameters

Consider the equation $y'' + y = \tan x$. What happens if we try to find a particular solution having the same *form* as the right hand side?

Suppose we guess that

$y_p = A\tan x$ then

$y_p' = A\sec^2 x$ ← we may have to match $\sec^2 x$

Try $y_p = A\tan x + B\sec^2 x$, then

$$y_p' = A\sec^2 x + 2B\sec^2 x\tan x$$ ← need to account for $\sec^2 x\tan x$

Try

$$y_p = A\tan x + B\sec^2 x + C\sec^2 x\tan x$$

$$y_p' = A\sec^2 x + 2B\sec^2 x\tan x + C\sec^4 x + 2\sec^2 x\tan^2 x$$

We can't account for all functions that come up when taking derivatives.

Consider the equation $x^2y'' + xy' - 4y = e^x$. What happens if we assume $y_p = Ae^x$?

$y_p' = Ae^x$, $y_p'' = Ae^x$

$x^2 y_p'' + x y_p' - 4y_p = e^x$

$x^2 Ae^x + x Ae^x - 4Ae^x = e^x$

We can't match due to xe^x and x^2e^x terms.

We need another method!

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x) = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

we'll substitute this into the ODE

$$y_p' = u_1 y_1' + u_2 y_2' + \underbrace{u_1' y_1 + u_2' y_2}_{\text{assume this sum is zero}}$$

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Remember that $\quad y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

We'll pick this up on Thursday.