March 15 Math 2306 sec 59 Spring 2016

Section 9: Method of Undetermined Coefficients

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply that term by x^n, where n is the smallest positive integer that eliminate the duplication.

Find the form of the particular soluition

$$y'' - 2y' + 5y = e^x + 7\sin(2x)$$

Find y_c: $\quad y'' - 2y' + 5y = 0$

Ch. eqn. $\quad m^2 - 2m + 5 = 0$

Complete the square $\quad m^2 - 2m + 1 - 1 + 5 = 0$

$$(m-1)^2 + 4 = 0$$

$$(m-1)^2 = -4 \Rightarrow m - 1 = \pm 2i$$

$\Rightarrow m = 1 \pm 2i \qquad \alpha \pm i\beta \qquad \alpha = 1,\ \beta = 2$

$$y_1 = e^x \cos(2x) \quad , \quad y_2 = e^x \sin(2x)$$

$$y_c = c_1 e^x \cos(2x) + c_2 e^x \sin(2x)$$

Let $g_1(x) = e^x$ $\qquad y_{p_1} = Ae^x$ $\qquad$ Doesn't match y_c will work as written

Let $g_2(x) = 7\sin(2x)$ $\qquad y_{p_2} = B\sin(2x) + C\cos(2x)$

will work doesn't match y_c

So $y_p = Ae^x + B\sin(2x) + C\cos(2x)$

Solve the IVP

$$y'' - 4y' + 4y = 8x - 4 \quad y(0) = 3, \quad y'(0) = -2$$

Find y_c: $y'' - 4y' + 4y = 0$, $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$ $m = 2$ repeated

$y_1 = e^{2x}$, $y_2 = xe^{2x}$

so $y_c = c_1 e^{2x} + c_2 x e^{2x}$

Find y_p: $g(x) = 8x - 4$, so $y_p = Ax + B$

$y_p' = A$, $y_p'' = 0$

We need
$$y_p'' - 4y_p' + 4y_p = 8x - 4$$
$$0 - 4(A) + 4(Ax+B) = 8x - 4$$
$$4Ax + (-4A + 4B) = 8x - 4$$

$$4A = 8 \implies A = 2$$
$$-4A + 4B = -4 \implies 4B = -4 + 4A = -4 + 8 = 4 \implies B = 1$$

$$y_p = 2x + 1$$

The general solution is

$$y = c_1 e^{2x} + c_2 x e^{2x} + 2x + 1$$

Apply $y(0) = 3$, $y'(0) = -2$

$$y' = 2c_1 e^{2x} + c_2 e^{2x} + 2c_2 x e^{2x} + 2$$

$$y(0) = c_1 e^0 + c_2 \cdot 0 \, e^0 + 2 \cdot 0 + 1 = 3 \Rightarrow c_1 + 1 = 3 \Rightarrow c_1 = 2$$

$$y'(0) = 2c_1 e^0 + c_2 e^0 + 2c_2 \cdot 0 \, e^0 + 2 = -2$$

$$c_2 = -2 - 2 - 2c_1 = -8$$

The solution to the IVP is

$$y = 2e^{2x} - 8xe^{2x} + 2x + 1.$$

Solve the IVP

$$y'' - y = 4e^{-x} \quad y(0) = -1, \quad y'(0) = 1$$

Find y_c : $y'' - y = 0$ $\quad m^2 - 1 = 0 \Rightarrow m = 1$ or $m = -1$

$y_1 = e^{x}$, $y_2 = e^{-x}$ so $y_c = c_1 e^{x} + c_2 e^{-x}$

Find y_p : $g(x) = 4e^{-x}$, $y_p = Ae^{-x}$ Duplicates $c_2 y_2$

Modify $y_p = (Ae^{-x})x = Axe^{-x}$ This will work.

$y_p = Axe^{-x}$

$y_p' = Ae^{-x} - Axe^{-x}$

$y_p'' = -Ae^{-x} - Ae^{-x} + Axe^{-x} = -2Ae^{-x} + Axe^{-x}$

we require

$$y_p'' - y_p = 4e^{-x}$$

$$-2Ae^{-x} + Axe^{-x} - Axe^{-x} = 4e^{-x}$$

$$-2Ae^{-x} = 4e^{-x}$$

$$-2A = 4 \Rightarrow A = -2$$

So $y_p = -2xe^{-x}$

The general solution is

$$y = c_1 e^{x} + c_2 e^{-x} - 2x e^{-x}$$

Apply $y(0) = -1$, $y'(0) = 1$

$$y' = c_1 e^{x} - c_2 e^{-x} - 2e^{-x} + 2x e^{-x}$$

$$y(0) = c_1 e^{0} + c_2 e^{0} - 2\cdot 0\, e^{0} = -1 \Rightarrow \quad c_1 + c_2 = -1$$

$$y'(0) = c_1 e^{0} - c_2 e^{0} - 2e^{0} + 2\cdot 0 e^{0} = 1 \Rightarrow \quad c_1 - c_2 = 3$$

add $\quad 2c_1 = 2 \Rightarrow c_1 = 1$

$c_2 = -1 - c_1 = -1 - 1 = -2$

The solution to the IVP is

$$\boxed{y = e^x - 2e^{-x} - 2xe^{-x}.}$$

Section 10: Variation of Parameters

Consider the equation $y'' + y = \tan x$. What happens if we try to find a particular solution having the same *form* as the right hand side?

Suppose we guess that

$y_p = A\tan x$ then

$y_p' = A\sec^2 x$ we may need to match $\sec^2 x$ terms

Try $y_p = A\tan x + B\sec^2 x$, then

$y_p' = A\sec^2 x + 2B\sec^2 x\tan x$ we may need to match $\sec^2 x\tan x$ terms

Try $y_p = A\tan x + B\sec^2 x + C\sec^2 x \tan x$

$$y_p' = A\sec^2 x + 2B\sec^2 x \tan x + C\sec^4 x + 2C\sec^2 x \tan^2 x$$

We can't account for every type of function that can arise due to differentiation.

Consider the equation $x^2y'' + xy' - 4y = e^x$. What happens if we assume $y_p = Ae^x$?

$$y_p' = Ae^x, \quad y_p'' = Ae^x$$

We need $$x^2 y_p'' + x y_p' - 4y_p = e^x$$

$$x^2(Ae^x) + x(Ae^x) - 4Ae^x = e^x$$

$$Ax^2e^x + Axe^x - 4Ae^x = e^x$$

We can't match to the Right w/ x^2e^x and xe^x terms.

We need another method!

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x) = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x)+u_2(x)y_2(x)$

Substitute into the ODE

$$y_p' = u_1 y_1' + u_2 y_2' + \underbrace{u_1' y_1 + u_2' y_2}_{\text{assume this is zero}}$$

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Remember that $\quad y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

We need $y_p'' + P(x)\,y_p' + Q(x)\,y_p = g(x)$

$$u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'' + P(x)\left(u_1y_1' + u_2y_2'\right) + Q(x)\left(u_1y_1 + u_2y_2\right) = g(x)$$

$$u_1'y_1' + u_2'y_2' + u_1\underbrace{\left(y_1'' + P(x)y_1' + Q(x)y_1\right)}_{\text{"}0\text{"}} + u_2\underbrace{\left(y_2'' + P(x)y_2' + Q(x)y_2\right)}_{\text{"}0\text{"}} = g(x)$$

Since y_1, y_2 solve the homogeneous equation.

We have 2 equations for u_1 and u_2

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = g(x)$$

We'll solve using Crammer's rule

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

Let $W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = 0 - gy_2 = -gy_2$

and $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = y_1 g - 0 = y_1 g$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

Then $u_1' = \dfrac{W_1}{W}$ and $u_2' = \dfrac{W_2}{W}$

We have the formulas

$$u_1(x) = \int \frac{-y_2(x)\, g(x)}{W}\, dx$$

and

$$u_2(x) = \int \frac{y_1(x)\, g(x)}{W}\, dx$$