

Section 11: Linear Mechanical Equations

Simple Harmonic Motion

For mass m attached to spring with spring constant k , in the absence of damping or driving force, the displacement x from equilibrium satisfies the second order equation

$$x'' + \omega^2 x = 0, \quad \text{where} \quad \omega^2 = \frac{k}{m}$$

Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

Total Force = Force of spring + Force of damping

$$m \frac{d^2 x}{dt^2} = -\beta \frac{dx}{dt} - kx \quad \implies \quad \frac{d^2 x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Damping Cases

$$\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = 0$$

- ▶ If there are two distinct real roots ($\lambda^2 > \omega^2$), the system is overdamped.
- ▶ If there is one repeated real root ($\lambda^2 = \omega^2$), the system is critically damped.
- ▶ If there are complex conjugate roots ($\lambda^2 < \omega^2$), the system is underdamped.

Comparison of Damping

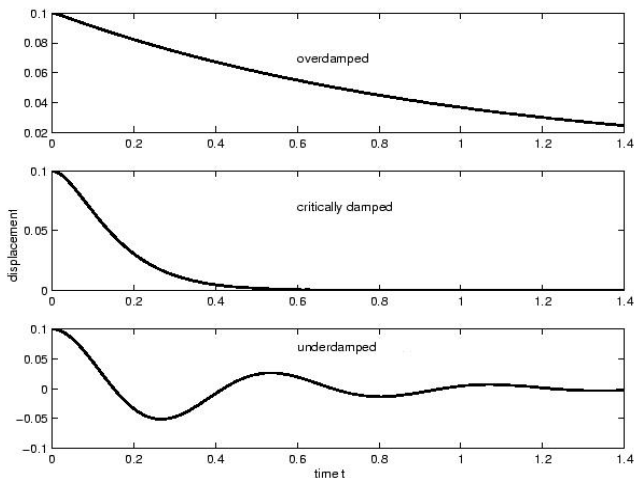


Figure: Comparison of motion for the three damping types.

A word on initial conditions

To determine the displacement of a mass in a spring-mass system (with or without damping), the ODE must be supplemented with initial conditions.

$$x(0) = x_0 \quad \text{where did the mass start?}$$

$$x'(0) = x_1 \quad \text{what was its starting velocity?}$$

Special Cases:

- ▶ If the mass starts **from equilibrium**, then $x(0) = 0$.
- ▶ If the mass starts **from rest**, then $x'(0) = 0$.

Example

A 3 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 12 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped. If the mass is released from the equilibrium position with an upward velocity of 1 m/sec, solve the resulting initial value problem.

$$mx'' + \beta x' + kx = 0 \quad m=3, \quad k=12, \quad \beta=12$$

$$3x'' + 12x' + 12x = 0$$

$$\Rightarrow x'' + 4x' + 4x = 0$$

$$\text{Charact. eqn} \quad r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0 \Rightarrow r = -2 \text{ repeated}$$

The system is critically damped.

$$x_1 = e^{-2t}, \quad x_2 = te^{-2t} \quad \text{so}$$

the general solution

$$x = c_1 e^{-2t} + c_2 t e^{-2t}$$

Initial conditions

$$x(0) = 0 \quad (\text{from equilibrium})$$

$$x'(0) = 1 \quad (\text{initial upward velocity of } 1 \frac{\text{m}}{\text{sec}})$$

$$x = c_1 e^{-2t} + c_2 t e^{-2t}$$

$$x' = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$$

$$x(0) = c_1 e^0 + c_2 \cdot 0 e^0 = 0 \Rightarrow c_1 = 0$$

$$x'(0) = c_2 e^0 - 2c_2 \cdot 0 e^0 = 1 \Rightarrow c_2 = 1$$

So the displacement

$$x = t e^{-2t}.$$

Example Continued...

For the spring-mass-damper system in the previous example, determine the maximum displacement of the mass.

$$x(t) = t e^{-2t}$$

Find critical #

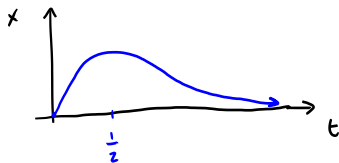
$$x'(t) = e^{-2t} - 2t e^{-2t}$$

$$= e^{-2t} (1 - 2t)$$

$$x'(t) = 0 \Rightarrow t = \frac{1}{2}$$

$$x'(0) = 1 > 0 \quad x'(1) < 0$$

(1st derivative test)



The maximum displacement is

$$x\left(\frac{1}{2}\right) = \frac{1}{2} e^{-2 \cdot \frac{1}{2}} = \frac{1}{2} e^{-1} = \frac{1}{2e}$$

i.e. $\frac{1}{2e}$ m

Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m \frac{d^2 x}{dt^2} = -\beta \frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2 x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = F(t)$$

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$.
Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and} \quad (2) \quad \gamma = \omega.$$

Taking the sine case, the DE is

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

with complementary solution

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$$x_p = A \cos(\gamma t) + B \sin(\gamma t) \quad \text{if } \gamma \neq \omega$$

Doesn't duplicate x_c , so this works.

The solution will look like

$$x = c_1 \cos(\omega t) + c_2 \sin(\omega t) + A \cos(\gamma t) + B \sin(\gamma t).$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$$x_p = A \cos(\gamma t) + B \sin(\gamma t) \quad \text{If } \gamma = \omega, \text{ this duplicates } x_c.$$

$$\text{Then } x_p = At \cos(\omega t) + Bt \sin(\omega t)$$

The solution will be

$$x = c_1 \cos(\omega t) + c_2 \sin(\omega t) + At \cos(\omega t) + Bt \sin(\omega t).$$

Forced Undamped Motion and Resonance

For $F(t) = F_0 \sin(\gamma t)$ starting from rest at equilibrium:

$$\text{Case (1): } x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2} \left(\sin(\gamma t) - \frac{\gamma}{\omega} \sin(\omega t) \right)$$

If $\gamma \approx \omega$, the amplitude of motion could be rather large!

Pure Resonance

Case (2): $x'' + \omega^2 x = F_0 \sin(\omega t)$, $x(0) = 0$, $x'(0) = 0$

$$x(t) = \frac{F_0}{2\omega^2} \sin(\omega t) - \frac{F_0}{2\omega} t \cos(\omega t)$$

Note that the amplitude, α , of the second term is a function of t :

$$\alpha(t) = \frac{F_0 t}{2\omega}$$

which grows without bound!

► Forced Motion and Resonance Applet

Choose "Elongation diagram" to see a plot of displacement. Try exciter frequencies close to ω .

Section 12: LRC Series Circuits

Potential Drops Across Components:

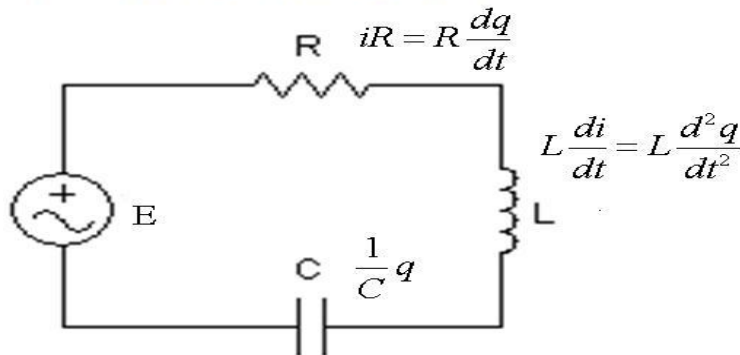


Figure: Kirchhoff's Law: The charge q on the capacitor satisfies $Lq'' + Rq' + \frac{1}{C}q = E(t)$.

This is a second order, linear, constant coefficient nonhomogeneous (if $E \neq 0$) equation.

LRC Series Circuit (Free Electrical Vibrations)

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = 0$$

If the applied force $E(t) = 0$, then the **electrical vibrations** of the circuit are said to be **free**. These are categorized as

overdamped if
critically damped if
underdamped if

$$R^2 - 4L/C > 0,$$

$$R^2 - 4L/C = 0,$$

$$R^2 - 4L/C < 0.$$

← 2 real roots
← one repeated root
↑ complex conjugate roots.

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

The function of q_c is influenced by the initial state (q_0 and i_0) and will decay exponentially as $t \rightarrow \infty$. Hence q_c is called the **transient state charge** of the system.

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From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

The function q_p is independent of the initial state but depends on the characteristics of the circuit (L , R , and C) and the applied voltage E . q_p is called the **steady state charge** of the system.