March 15 Math 2335 sec 51 Spring 2016

Section 4.3: Interpolating Using Spline Functions

We've seen that when we can choose our nodes, we may be able to use a high degree polynomial interpolation and reduce the wiggly effect near the ends. But we don't always have an option to pick our nodes.

An alternative is to fit data with a piece-wise defined function that interpolates. We'll consider building a function whose pieces are polynomial.

Motivating Example

Consider the data set

x	0	$\frac{1}{2}$	1	2	$\frac{5}{2}$	4
y	1	-1	1	1	$\frac{3}{2}$	2

A simple way to interpolate is to plot the points and connect each adjacent pair with a straight line segment.

Take straight line connecting $(0,1)$ to $(\frac{1}{2},-1)$.

Then another connecting $(\frac{1}{2},-1)$ to $(1,1)$.

And so forth.

Piece-wise Linear Interpolation

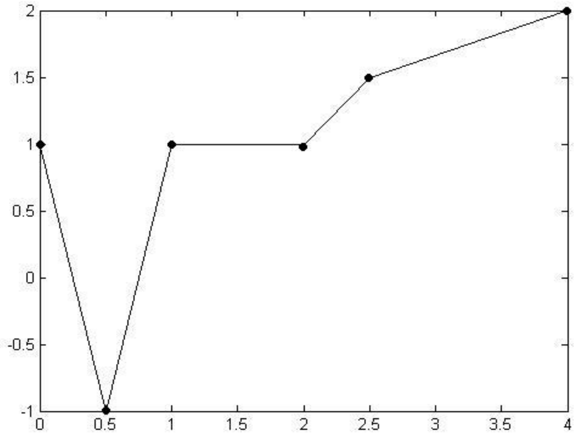

Figure: It does interpolate, but the graph is not very smooth looking.

Interpolation of the Same Data with $P_5(x)$

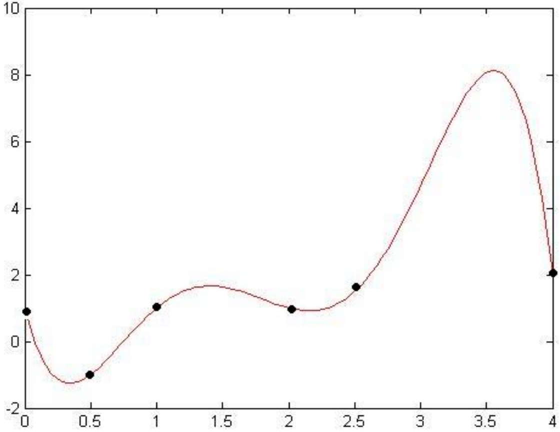

Figure: The graph of $P_5(x)$ is significantly smoother, but how does the overall shape compare?

Comparison: Piece-wise Linear -vs- $P_5(x)$

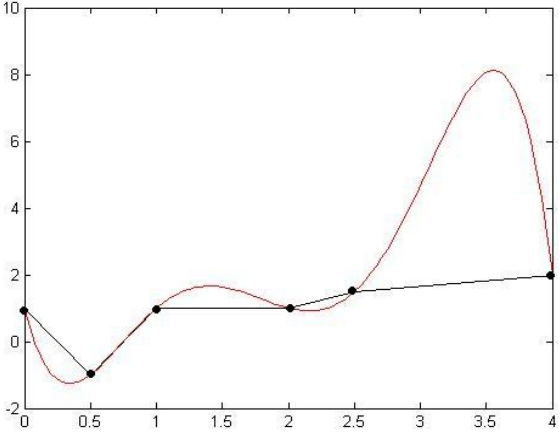

Figure: Both pass through the data, but the curves are very different between the data points.

Piece-wise Cubic Interpolation

An alternative is to take each successive pair of points and fit them with a piece of (polynomial) curve $s(x)$ that

- passes through each data point,
- is continuous on the entire interval (the pieces are connected), and
- such that $s'(x)$ and $s''(x)$ are continuous on the entire interval (they connect without kinks or corners).

The last condition suggests we will need the pieces to be cubic, or up to degree three.

Piece-wise Cubic Interpolation

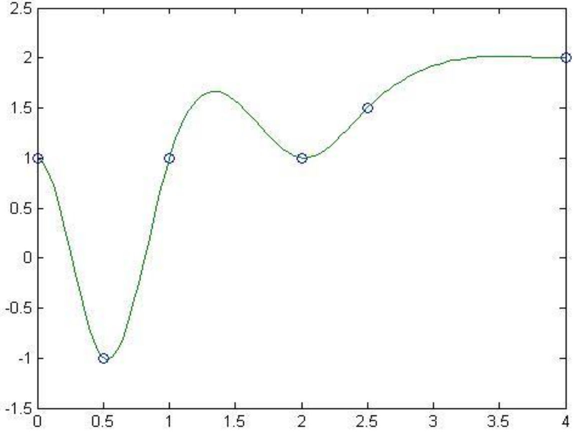

Figure: The data is interpolated by five pieces of cubic function.

Comparison: Piece-wise Linear -vs- Piece-wise Cubic

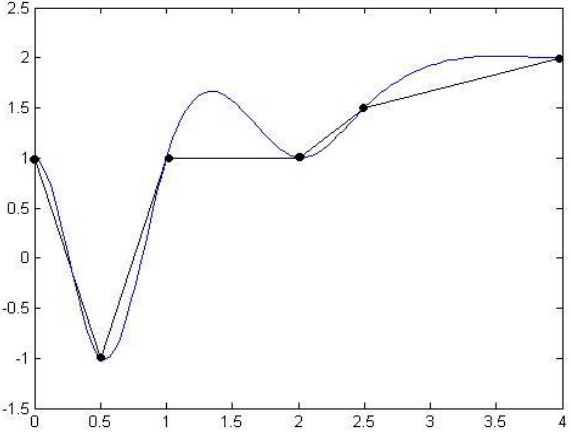

Figure: The overall shape is preserved with piecewise cubic and the kinks are smoothed out.

Natural Cubic Spline

Suppose that the n data points (x_i, y_i), $j = 1, 2, \dots, n$ are given. Assume $x_1 = a$, $x_{j-1} < x_j$ and $x_n = b$. There exists a function $s(x)$ that interpolates these points—i.e.

$$s(x_j) = y_j, \quad \text{for} \quad j = 1, \dots, n$$

satisfying the following properties:

S1. $s(x)$ is a polynomial of degree ≤ 3 on each interval $[x_j, x_{j+1}]$, $j = 1, \dots n-1$.

S2. $s(x)$, $s'(x)$, and $s''(x)$ are continuous on $[a, b]$.

S3. $s''(x_1) = 0$ and $s''(x_n) = 0$ [1]

The curve $s(x)$ is called the *natural cubic spline* that interpolates the data.

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[1] This condition may be changed leading to other cubic splines.

Example

Consider the function $s(x)$ given. Determine if it is a cubic spline on $[1,3]$ (does it satify S1 and S2). Is it a natural cubic spline (does it satisfy S3)?

$$s(x) = \begin{cases} 2x^3 - 6x^2 + 8x - 5, & 1 \le x \le 2 \\ -2x^3 + 18x^2 - 40x + 27, & 2 \le x \le 3 \end{cases}$$

S1 is satisfied, both pieces are polynomials of degree at most 3.

$$S'(x) = \begin{cases} 6x^2 - 12x + 8, & 1 \le x < 2 \\ -6x^2 + 36x - 40, & 2 < x \le 3 \end{cases}$$

$$s''(x) = \begin{cases} 12x - 12, & 1 \le x < 2 \\ -12x + 36, & 2 < x \le 3 \end{cases}$$

Continuity of S:

$$\lim_{x\to 2^-} S(x) = \lim_{x\to 2^-} (2x^3 - 6x^2 + 8x - 5)$$

$$= 2\cdot 8 - 6\cdot 4 + 8\cdot 2 - 5 \quad = 3$$

S is continuous @ 2

$$\lim_{x\to 2^+} S(x) = \lim_{x\to 2^+} \left(-2x^3 + 18x^2 - 40x + 27\right)$$

$$= -2\cdot 8 + 18\cdot 4 - 40\cdot 2 + 27 \quad = 3$$

Check continuity of s':

$$\lim_{x\to 2^-} s'(x) = \lim_{x\to 2^-} \left(6x^2 - 12x + 8\right) =$$

$$6\cdot 4 - 12\cdot 2 + 8 = 8$$

s' is continuous @ 2

$$\lim_{x\to 2^+} s'(x) = \lim_{x\to 2^+} \left(-6x^2 + 36x - 40\right)$$

$$= -6\cdot 4 + 36\cdot 2 - 40 = 8$$

Check continuity of s'':

$$\lim_{x\to 2^-} s''(x) = \lim_{x\to 2^-} \left(12x - 12\right) = 24 - 12 = 12$$

$$\lim_{x\to 2^+} S''(x) = \lim_{x\to 2^+} (-12x+36) = -24+36 = 12$$

S, S', S'' are all continuous on $[1,3]$.

Is it natural?

$$S''(1) = 12\cdot 1 - 12 = 0, \quad S''(3) = -12\cdot 3 + 36 = 0$$

S3 holds. This is a natural cubic spline.

Example

Consider the function $s(x)$ given. Determine if it is a cubic spline on $[0,2]$ (does it satify S1 and S2). Is it a natural cubic spline (does it satisfy S3)?

$$s(x) = \begin{cases} (x-1)^3, & 0 \le x \le 1 \\ -2(x-1)^2, & 1 \le x \le 2 \end{cases}$$

S1 holds, each piece is a polynomial of degree at most 3.

$$s'(x) = \begin{cases} 3(x-1)^2, & 0 \le x < 1 \\ -4(x-1), & 1 < x \le 2 \end{cases}$$

$$s''(x) = \begin{cases} 6(x-1), & 0 \le x < 1 \\ -4, & 1 < x \le 2 \end{cases}$$

Is $s(x)$ continuous?

$$\lim_{x \to 1^-} s(x) = \lim_{x \to 1^-} (x-1)^3 = 0$$

s is continuous

$$\lim_{x \to 1^+} s(x) = \lim_{x \to 1^+} -2(x-1)^2 = 0$$

How about s'? $\lim_{x \to 1^-} s'(x) = \lim_{x \to 1^-} 3(x-1)^2 = 0$

$$\lim_{x\to 1^+} S'(x) = \lim_{x\to 1^+} -4(x-1) = 0$$

S' is continuous

How about S''?

$$\lim_{x\to 1^-} S''(x) = \lim_{x\to 1^-} 6(x-1) = 0$$

$$\lim_{x\to 1^+} S''(x) = \lim_{x\to 1^+} -4 = -4$$

S'' is not continuous.

S is <u>not</u> a cubic spline.

Construction of the Natural Cubic Spline

(1) Start with the data $(x_1, y_1), \dots, (x_n, y_n)$ with $a = x_1 < x_2 < \cdots < x_n = b$. Set

$$h_j = x_{j+1} - x_j \quad \text{(the length of the } j^{th} \text{ subinterval).}$$

(2) Define the numbers $M_1, \dots, M_n$ by

$$M_1 = s''(x_1), \quad M_2 = s''(x_2), \quad M_3 = s''(x_3), \dots, M_n = s''(x_n).$$

Question: What are the values of M_1 and M_n?

By definition of the natural cubic spline $M_1 = M_n = 0$.

Construction of the Natural Cubic Spline

(3) Consider the subinterval $x_j \le x \le x_{j+1}$. We have $M_j = s''(x_j)$ and $M_{j+1} = s''(x_{j+1})$, and on this interval, $s''(x)$ must be a line. So it can be written as

$$s''(x) = \frac{(x_{j+1} - x)M_j + (x - x_j)M_{j+1}}{x_{j+1} - x_j}, \quad \text{for} \quad x_j \le x \le x_{j+1}.$$

Question: Why do we know that $s''(x)$ must be a line on this interval?

Since s is at most cubic, s' is quadratic and

s'' linear.

Construction of the Natural Cubic Spline

(4) To get $s(x)$ on this subinterval, we integrate the line $s''(x)$ twice and impose the conditions

$$s(x_j) = y_j, \quad \text{and} \quad s(x_{j+1}) = y_{j+1}.$$

With a little patience, we obtain the formula for $s(x)$ on $[x_j, x_{j+1}]$

$$s(x) = \frac{M_j}{6h_j}(x_{j+1} - x)^3 + \frac{M_{j+1}}{6h_j}(x - x_j)^3 + \frac{y_j}{h_j}(x_{j+1} - x) + \frac{y_{j+1}}{h_j}(x - x_j) -$$

$$-\frac{h_j}{6}\left[M_j(x_{j+1} - x) + M_{j+1}(x - x_j)\right], \quad x_j \leq x \leq x_{j+1}$$

Recall that $h_j = x_{j+1} - x_j$.

Construction of the Natural Cubic Spline

(5) Finally, we impose the condition that $s'(x)$ is continuous (match the equations from $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$) to get a system of equations for the numbers M_j. We end up with

$$M_1 = M_n = 0, \quad \text{and}$$

$$\frac{h_{j-1}}{6} M_{j-1} + \frac{h_j + h_{j-1}}{3} M_j + \frac{h_j}{6} M_{j+1} = \frac{y_{j+1} - y_j}{h_j} - \frac{y_j - y_{j-1}}{h_{j-1}}$$

$$\text{for} \quad j = 2, \ldots, n-1.$$

Spline Example 1

Consider the set of data $\{(0,1),(1,1),(2,3)\}$.

(a) Find the piece-wise linear interpolation function for this data, and

(b) Find the natural cubic spline that interpolates this data.

(a) The line through $(0,1)$, $(1,1)$ slope $m = \frac{1-1}{1-0} = 0$

so $y = 1$

The line through $(1,1)$, $(2,3)$ slope $m = \frac{3-1}{2-1} = 2$

$y - 1 = 2(x-1) = 2x - 2 \Rightarrow y = 2x - 1$

The piece wise linear interpolating function is

$$\ell(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 2x-1, & 1 \le x \le 2 \end{cases}$$

For the cubic spline: here $h_j = h = 1$ for all j

$$M_1 = 0, \quad M_2 = ?, \quad M_3 = 0$$

$j=2$ $$\underbrace{\cancel{M_1}}_{0} + 4M_2 + \underbrace{\cancel{M_3}}_{0} = \frac{6}{1^2}(y_3 - 2y_2 + y_1)$$

$$4M_2 = 6(3 - 2\cdot 1 + 1) = 12$$

$$\boxed{M_2 = 3}$$

Construct $S(x)$: use $h_j = h = 1$

$j=1$ $M_1 = 0$

$$S(x) = \frac{\overset{=0}{M_1}}{6h}(x_2 - x)^3 + \frac{M_2}{6h}(x - x_1)^3 + \frac{y_1}{h}(x_2 - x)$$

$$+ \frac{y_2}{h}(x - x_1) - \frac{h}{6}\left[\overset{=0}{M_1}(x_2 - x) + M_2(x - x_1)\right]$$

$$= \frac{3}{6}(x-0)^3 + \frac{1}{1}(1-x) + \frac{1}{1}(x-0) - \frac{1}{6}\left[3(x-0)\right]$$

$$= \frac{1}{2}x^3 + 1 - x + x - \frac{1}{2}x$$

$$= \frac{1}{2}x^3 - \frac{1}{2}x + 1$$

when $j = 2$ $\qquad M_3 = 0$

$$S(x) = \frac{M_2}{6h}(x_3 - x)^3 + \frac{\overset{=0}{M_3}}{6h}(x - x_2)^3 + \frac{y_2}{h}(x_3 - x)$$

$$+ \frac{y_3}{h}(x - x_2) - \frac{h}{6}\left[M_2(x_3 - x) + \overset{=0}{M_3}(x - x_2)\right]$$

$$= \frac{3}{6}(2 - x)^3 + \frac{1}{1}(2 - x) + \frac{3}{1}(x - 1) - \frac{1}{6}\left[3(2 - x)\right]$$

$$= \frac{1}{2}\left(8 - 3\cdot 4x + 3\cdot 2x^2 - x^3\right) + 2 - x + 3x - 3 - 1 + \frac{1}{2}x$$

$$= -\frac{1}{2}x^3 + 3x^2 + \left(-6 - 1 + 3 + \frac{1}{2}\right)x + \left(4 + 2 - 3 - 1\right)$$

$$= -\frac{1}{2}x^3 + 3x^2 - \frac{7}{2}x + 2$$

Hence

$$S(x) = \begin{cases} \frac{1}{2}x^3 - \frac{1}{2}x + 1, & 0 \le x \le 1 \\ -\frac{1}{2}x^3 + 3x^2 - \frac{7}{2}x + 2, & 1 \le x \le 2 \end{cases}.$$

Example 1 Results

We found the piece-wise linear funtion $\ell(x)$ and the natural cubic spline function $s(x)$ that interpolate the data $\{(0,1),(1,1),(2,3)\}$.

$$\ell(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 2x-1, & 1 \le x \le 2 \end{cases}$$

$$s(x) = \begin{cases} \frac{1}{2}x^3 - \frac{1}{2}x + 1, & 0 \le x \le 1 \\ -\frac{1}{2}x^3 + 3x^2 - \frac{7}{2}x + 2, & 1 \le x \le 2 \end{cases}$$