March 16 Math 1190 sec. 62 Spring 2017

Section 4.2: Maximum and Minimum Values; Critical Numbers

We defined both local (relative) and absolute (global) extrema, and stated a few key definitions and theorems.

Theorem: Extreme Value Theorem (EVT) Suppose f is continuous on a closed interval $[a,b]$. Then f attains an absolute maximum value $f(d)$ and f attains an absolute minimum value $f(c)$ for some numbers c and d in $[a,b]$.

Related Theorems

Fermat's Theorem: If f has a local extremum at c and if $f'(c)$ exists, then

$$f'(c) = 0.$$

Definition: A **critical number** (a.k.a. critical point) of a function f is a number c in its domain such that either

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

Theorem: If f has a local extremum at c, then c is a critical number of f.

Example

Find all of the critical numbers of the function.

$$g(t) = t^{1/5}(12-t)$$

We did this on Tuesday and found that g has two critical numbers, 2 and 0.

Example

Find all of the critical numbers of the function.

$$F(x) = \frac{\ln x}{x}$$

The domain of F is $(0, \infty)$.

$$F'(x) = \frac{\left(\frac{d}{dx}\ln x\right)x - \ln x\left(\frac{d}{dx}x\right)}{x^2} = \frac{\frac{1}{x}\cdot x - \ln x\,(1)}{x^2}$$

$$= \frac{1-\ln x}{x^2}$$

$F'(x) = 0$ if the numerator is zero.

$$1 - \ln x = 0 \Rightarrow \ln x = 1 \Rightarrow e^{\ln x} = e^{1} \Rightarrow x = e.$$

$F'(x)$ is undefined if the denominator is zero.

$x^2 = 0 \Rightarrow x = 0$ which is not in the domain

So zero is not a crit. #

F has one critical number, e.

Question

Find the derivative of $f(x) = xe^x$.

(a) $f'(x) = xe^x$

(b) $f'(x) = e^x$

(c) $f'(x) = (x+1)e^x$

(d) $f'(x) = e^x + x^2 e^{x-1}$

$$f'(x) = \left(\frac{d}{dx} x\right) e^x + x \left(\frac{d}{dx} e^x\right)$$

$$= e^x + x e^x$$

$$= e^x (1+x)$$

Question

Find all of the critical numbers of the function.

$$f(x) = xe^x$$

$f'(x) = (x+1)e^x$

$f'(x) = 0 \Rightarrow (x+1)e^x = 0$

$x+1=0$ or $e^x = 0$

$x=-1$ no soln.

$f'(x)$ undefined never.

(a) -1 and 0

(b) -1

(c) -1 and e

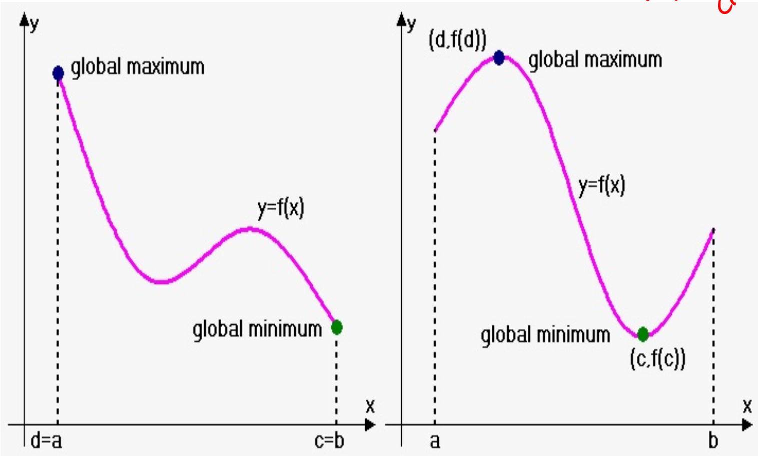
(d) There are none

Using the Extreme Value Theorem

When the EVT applies, each absolute extrema occurs either at

- at an end point, or
- in between at a critical point.

recall, local extrema have to happen @ critical numbers.



Example

Find the absolute maximum and absolute minimum values of the function on the closed interval.

(a) $g(t) = t^{1/5}(12 - t)$, on $[-1, 1]$

g is continuous
$[-1,1]$ is closed.

Extrema occur either @ an end point, or at a critical number between them.

g had two crit. #'s 0 and 2. Zero is on our interval. We'll check the three numbers,

$g(-1)$, $g(0)$, and $g(1)$.

$g(t) = t^{1/5}(12-t)$

$g(-1) = (-1)^{1/5}(12-(-1)) = (-1)(13) = -13$

$g(0) = (0)^{1/5}(12-0) = 0 \cdot 12 = 0$

$g(1) = (1)^{1/5}(12-1) = (1)(11) = 11$

abs. max value is $11 = g(1)$

abs. min value is $-13 = g(-1)$.

(b) $f(x) = xe^x$, on $[-3, 1]$

f is continuous
$[-3,1]$ is closed.

f had one critical number -1.

Check f @ end points and at the critical number between them.

$e \approx 2.71828\ldots$

e is close to 3

$f(-3) = -3e^{-3} = \frac{-3}{e^3}$

so $\frac{-3}{e^3} \sim \frac{-3}{27} = \frac{-1}{9}$

$f(-1) = -1e^{-1} = \frac{-1}{e}$ $\leftarrow$ min

and $\frac{-1}{e} \sim \frac{-1}{3}$

$f(1) = 1 \cdot e^1 = e$ $\leftarrow$ clearly our max.

The abs. max value is $e = f(1)$.

The abs. min value is $\frac{-1}{e} = f(-1)$.

Question

Find all of the critical numbers of the function

$$f(x) = 1+27x-x^3$$

$f'(x) = 27 - 3x^2 = 3(9-x^2)$

$f'(x) = 0 \Rightarrow 3(9-x^2) = 0 \Rightarrow 9 = x^2$

$x = \pm 3$

(a) 0 and 27

(b) 0 and 3

$f'(x)$ is never undefined.

(c) -3 and 3

(d) -3, 0, and 3

Question

crit #'s were 3 and -3

Find the absolute maximum and absolute minimum values of the function on the closed interval.

$f(0) = 1$ ← min

$f(3) = 55$ ← max

$f(4) = 45$

$$f(x) = 1 + 27x - x^3, \quad \text{on} \quad [0, 4]$$

(a) Minimum value is 1, maximum value is 55

(b) Minimum value is 1, maximum value is 35

(c) Minimum value is -53, maximum value is 55

(d) Minimum value is -53, maximum value is 35

Section 4.3: The Mean Value Theorem

Rolle's Theorem: Let f be a function that is

i continuous on the closed interval $[a, b]$,

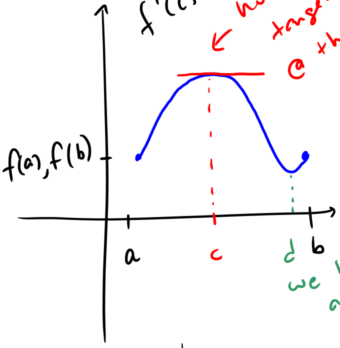
ii differentiable on the open interval (a, b), and

iii such that $f(a) = f(b)$.

Then there exists a number c in (a, b) such that $f'(c) = 0$.

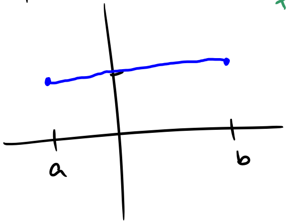
The Mean Value Theorem (MVT) is arguably the most significant theorem in calculus. This is even accounting for a theorem we'll discuss later called the *Fundamental Theorem of Calculus*.

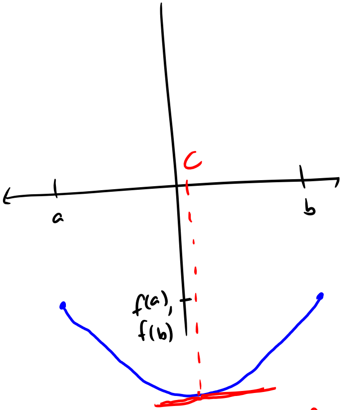
Rolle's Theorem



$f'(c)=0$ ← horizontal tangent @ the turning place

we have another $f'(d)=0$





$f'(c)=0$ horizontal tangent

Example

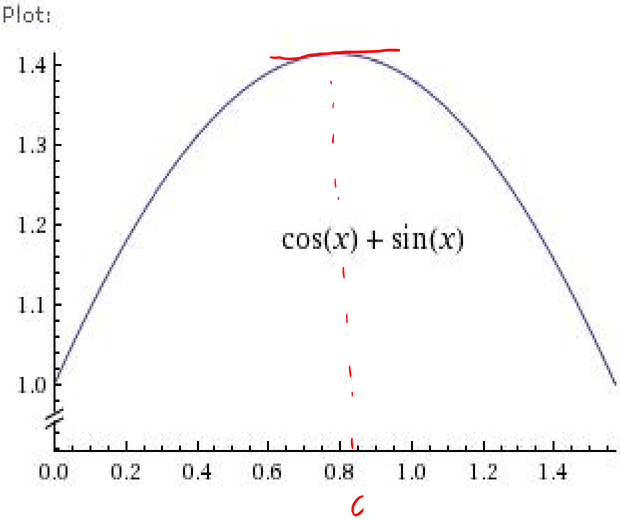
Show that the function $f(\theta) = \cos\theta + \sin\theta$ has at least one point c in $\left[0, \frac{\pi}{2}\right]$ such that $f'(c) = 0$.

f is continuous everywhere, so it is on $[0, \pi/2]$.

f is differentiable everywhere, so it is on $(0, \frac{\pi}{2})$.

$$\left.\begin{array}{l} f(0) = \cos 0 + \sin 0 = 1 + 0 = 1 \\ f(\frac{\pi}{2}) = \cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 0 + 1 = 1 \end{array}\right\} \Rightarrow f(0) = f(\tfrac{\pi}{2})$$

By Rolle's theorem, there must be some c in $(0, \frac{\pi}{2})$ such that $f'(c) = 0$.



Figure

The Mean Value Theorem

Theorem: Suppose f is a function that satisfies

i f is continuous on the closed interval $[a, b]$, and
ii f is differentiable on the open interval (a, b).

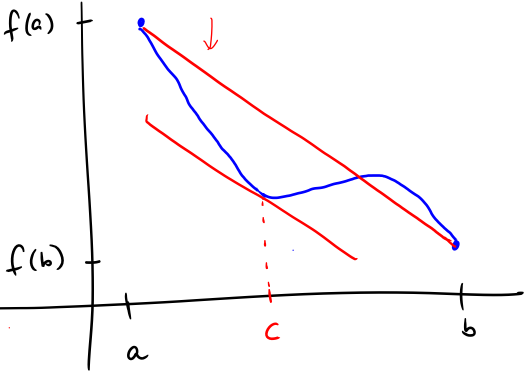
Then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad \text{equivalently} \quad f(b) - f(a) = f'(c)(b - a).$$

$\frac{f(b)-f(a)}{b-a}$ is the slope of the secant line connecting $(a, f(a))$ and $(b, f(b))$.

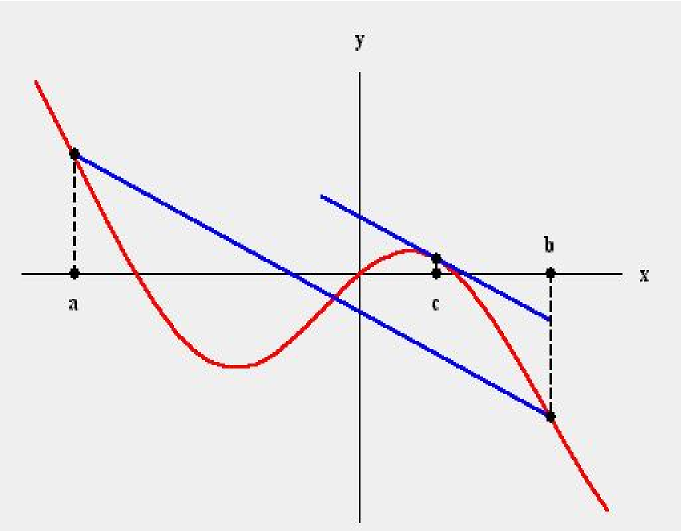
$$\frac{f(b)-f(a)}{b-a} = \text{slope of secant line}$$

slope

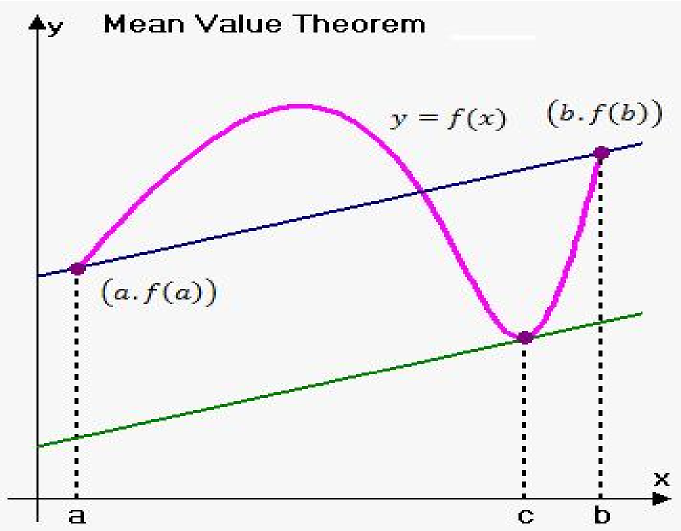


$f'(c)$ = the same slope

The tangent line is parallel to the secant line!



Figure



Figure

Figure: Celebration of the MVT in Beijing.

Example

Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all values of c that satisfy the conclusion of the MVT.

$$f(x) = x^3 - 2x, \quad [-2, 2]$$

f is continuous and differentiable on $[-2,2]$

(it's a polynomial!).

$$\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(-2)}{2-(-2)} = \frac{4-(-4)}{4} = 2$$

$$f(-2) = (-2)^3 - 2(-2) = -4 \qquad f(2) = 2^3 - 2(2) = 4$$

$$f(x) = x^3 - 2x \quad \Rightarrow \quad f'(x) = 3x^2 - 2$$

$$f'(c) = 2 \quad \Rightarrow \quad 3c^2 - 2 = 2$$

$$3c^2 = 4$$

$$c^2 = \frac{4}{3}$$

$$c = \sqrt{\frac{4}{3}} \quad \text{or} \quad c = -\sqrt{\frac{4}{3}}$$

$$= \frac{2}{\sqrt{3}} \quad \text{or} \quad c = \frac{-2}{\sqrt{3}}$$

These are the two solutions.