March 16 Math 1190 sec. 63 Spring 2017

Section 4.2: Maximum and Minimum Values; Critical Numbers

We defined both local (relative) and absolute (global) extrema, and stated a few key definitions and theorems.

Theorem: Extreme Value Theorem (EVT) Suppose f is continuous on a closed interval $[a, b]$. Then f attains an absolute maximum value $f(d)$ and f attains an absolute minimum value $f(c)$ for some numbers c and d in $[a, b]$.

Related Theorems

Fermat's Theorem: If f has a local extremum at c and if $f'(c)$ exists, then

$$f'(c) = 0.$$

Definition: A **critical number** (a.k.a. critical point) of a function f is a number c in its domain such that either

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

Theorem: If f has a local extremum at c, then c is a critical number of f.

Example

Find all of the critical numbers of the function.

$$g(t) = t^{1/5}(12-t)$$

We did this on Tuesday and found that g has two critical numbers, 2 and 0.

Example

Find all of the critical numbers of the function.

$$F(x) = \frac{\ln x}{x}$$

The domain of F is $(0, \infty)$.

Find $F'(x)$: $F'(x) =$

Question

Find the derivative of $f(x) = xe^x$.

(a) $f'(x) = xe^x$

(b) $f'(x) = e^x$

(c) $f'(x) = (x + 1)e^x$

(d) $f'(x) = e^x + x^2 e^{x-1}$

$f'(x) = x\left(\frac{d}{dx}e^x\right) + \left(\frac{d}{dx}x\right)e^x$

$= xe^x + 1e^x$

$= e^x(x+1).$

Question

Find all of the critical numbers of the function.

$$f(x) = xe^x$$

$f'(x) = (x+1)e^x$

$f'(x)$ is undefined never

$f'(x) = 0 \Rightarrow (x+1)e^x = 0$

$\Rightarrow x+1 = 0$ or $e^x = 0$

$x = -1$ no soln.

(a) -1 and 0

(b) -1

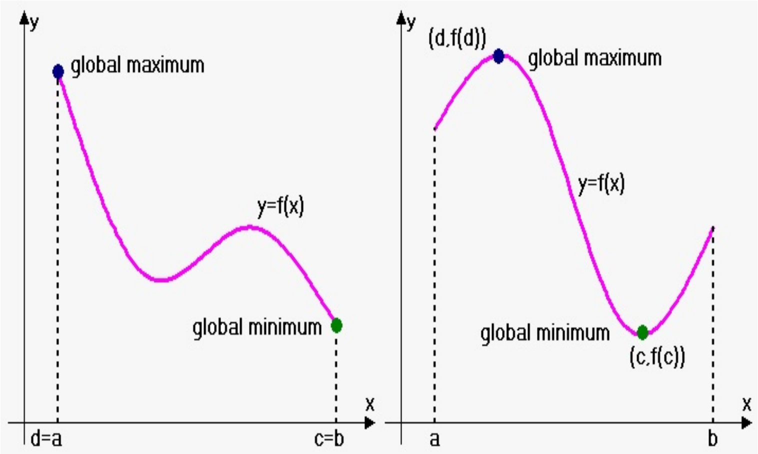
(c) -1 and e

(d) There are none

Using the Extreme Value Theorem

When the EVT applies, each absolute extrema occurs either at

- at an end point, or
- in between at a critical point.



Example

Find the absolute maximum and absolute minimum values of the function on the closed interval.

g is continuous

(a) $g(t) = t^{1/5}(12-t)$, on $[-1, 1]$ $[-1,1]$ is closed.

The extreme values occur either at end points or at critical point(s) between the ends.

g had critical numbers 0 and 2.

2 is outside of $[-1,1]$. So there's one critical number inside the interval.

we'll check $g(-1)$, $g(0)$, and $g(1)$.

$$g(t) = t^{1/5}(12-t)$$

$$g(-1) = (-1)^{1/5}(12-(-1)) = -1(13) = -13$$

$$g(0) = 0^{1/5}(12-0) = 0\cdot 12 = 0$$

$$g(1) = 1^{1/5}(12-1) = 1\cdot 11 = 11$$

The abs. max is $11 = g(1)$.

The abs. min is $-13 = g(-1)$.

(b) $f(x) = xe^x$, on $[-3, 1]$

f had one critical no. -1 which is in the interval. Compare $f(-3)$, $f(-1)$, $f(1)$.

$f(-3) = -3e^{-3} = \frac{-3}{e^3}$

$f(-1) = -1e^{-1} = \frac{-1}{e} \leftarrow \text{min}$

$f(1) = 1 \cdot e^1 = e \leftarrow \text{max}$

$e \approx 2.71828\ldots$

e is "close" to 3

$\left.\begin{array}{l} \frac{-3}{e^3} \sim \frac{-3}{3^3} = \frac{-1}{9} \\ \frac{-1}{e} \sim \frac{-1}{3} \end{array}\right\} \frac{-1}{e} < \frac{-3}{e^3}$

The absolute max value is $e = f(1)$.

The absolute min value is $\frac{-1}{e} = f(-1)$.

Question

Find all of the critical numbers of the function

$$f(x) = 1+27x-x^3$$

$f'(x) = 27 - 3x^2 = 3(9-x^2)$

$= 3(3-x)(3+x)$

(a) 0 and 27

$f'(x) = 0 \Rightarrow 3(3-x)(3+x) = 0$

$3-x=0$ or $3+x=0$

(b) 0 and 3

$x=3$ or $x=-3$

(c) -3 and 3

$f'(x)$ is always defined

(d) -3, 0, and 3

Question

Find the absolute maximum and absolute minimum values of the function on the closed interval.

f had one crit. number inside the interval.

$$f(x) = 1+27x-x^3, \quad \text{on} \quad [0,4]$$

$f(0) = 1$

$f(3) = 55$

$f(4) = 45$

(a) Minimum value is 1, maximum value is 55

(b) Minimum value is 1, maximum value is 35

(c) Minimum value is -53, maximum value is 55

(d) Minimum value is -53, maximum value is 35

Section 4.3: The Mean Value Theorem

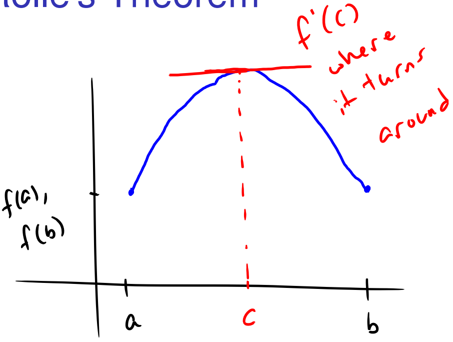
Rolle's Theorem: Let f be a function that is

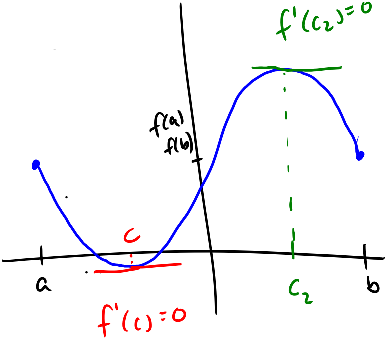
i continuous on the closed interval $[a, b]$,
ii differentiable on the open interval (a, b), and
iii such that $f(a) = f(b)$.

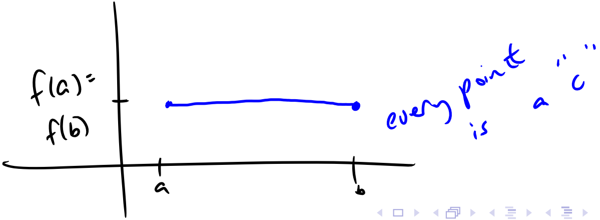
Then there exists a number c in (a, b) such that $f'(c) = 0$.

The Mean Value Theorem (MVT) is arguably the most significant theorem in calculus. This is even accounting for a theorem we'll discuss later called the *Fundamental Theorem of Calculus*.

Rolle's Theorem







Example

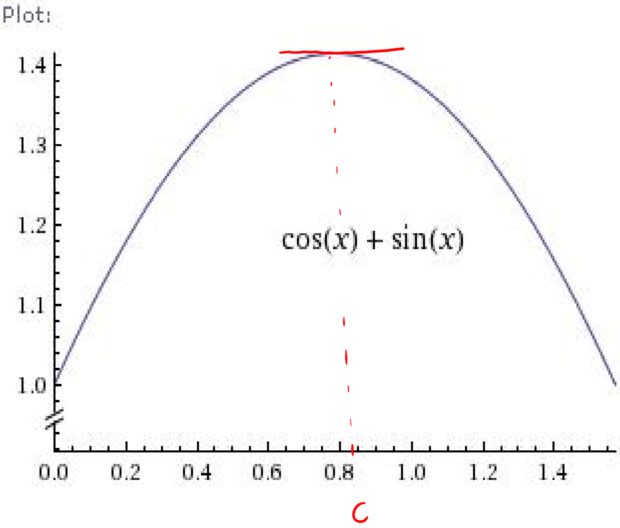
Show that the function $f(\theta) = \cos\theta + \sin\theta$ has at least one point c in $\left[0, \frac{\pi}{2}\right]$ such that $f'(c) = 0$.

f is continuous and differentiable everywhere.

So it's continuous on $[0, \pi/2]$ and differentiable on $\left(0, \frac{\pi}{2}\right)$.

$$\left.\begin{aligned} f(0) &= \cos 0 + \sin 0 = 1 + 0 = 1 \\ f\left(\tfrac{\pi}{2}\right) &= \cos\tfrac{\pi}{2} + \sin\tfrac{\pi}{2} = 0 + 1 = 1 \end{aligned}\right\} f(0) = f(\pi/2)$$

By Rolle's Thm, there exists c in $(0, \pi/2)$ such that $f'(c) = 0$.



Figure

The Mean Value Theorem

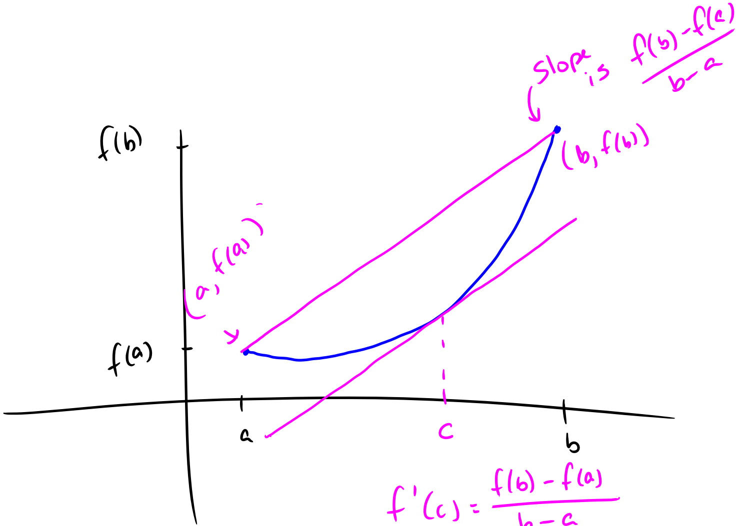
Theorem: Suppose f is a function that satisfies

i f is continuous on the closed interval $[a, b]$, and
ii f is differentiable on the open interval (a, b).

Then there exists a number c in (a, b) such that

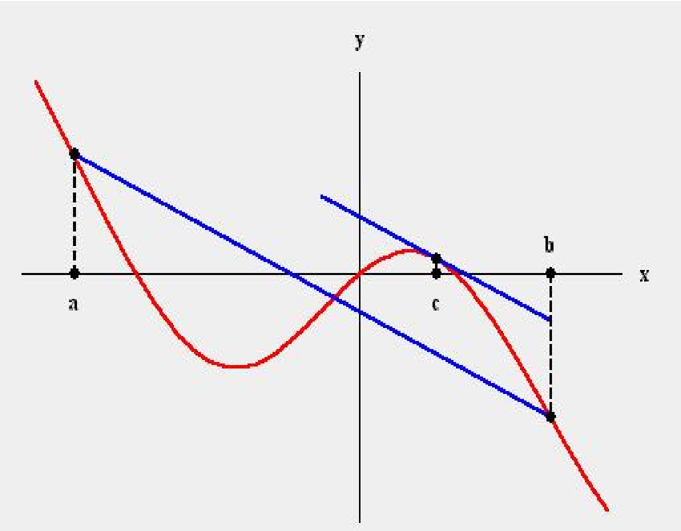
$$f'(c) = \underbrace{\frac{f(b) - f(a)}{b - a}}, \quad \text{equivalently} \quad f(b) - f(a) = f'(c)(b - a).$$

This is the slope of the secant line through $(a, f(a))$ and $(b, f(b))$.

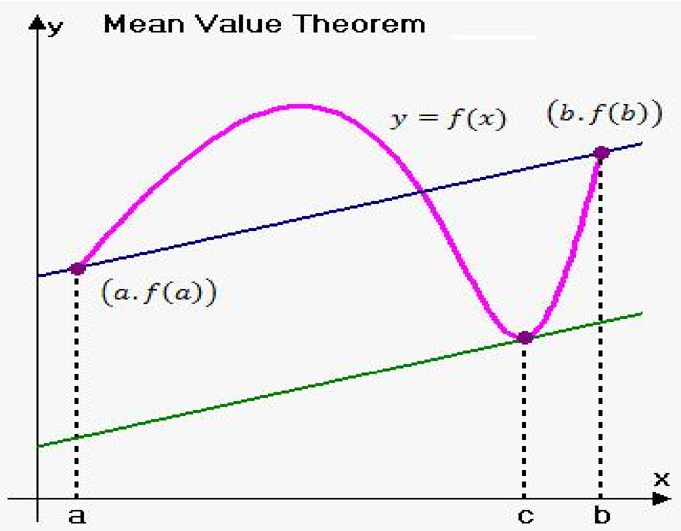


$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

The tangent @ c is parallel to the secant line!



Figure



Figure

Figure: Celebration of the MVT in Beijing.

Example

Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all values of c that satisfy the conclusion of the MVT.

$$f(x) = x^3 - 2x, \quad [-2, 2]$$

As a polynomial, f is continuous on $[-2,2]$ and differentiable on $(-2,2)$.

$f(-2) = (-2)^3 - 2(-2) = -4$, $f(2) = 2^3 - 2 \cdot 2 = 4$

$$\frac{f(2) - f(-2)}{2-(-2)} = \frac{4-(-4)}{2-(-2)} = 2$$

$$f(x) = x^3 - 2x \implies f'(x) = 3x^2 - 2$$

we need $f'(c) = 2$

$$3c^2 - 2 = 2 \implies 3c^2 = 4$$

$$c^2 = \frac{4}{3}$$

$$c = \pm\sqrt{\frac{4}{3}}$$

$c = \dfrac{2}{\sqrt{3}}$ or $c = \dfrac{-2}{\sqrt{3}}$

Both are in the interval. These are the two answers.