Mar. 16 Math 2254H sec 015H Spring 2015

Section 11.1: Sequences

Recall: A sequence is an ordered list of numbers. More generally, a sequence is a function

$$a_n = f(n)$$

whose domain is a subset of the integers.

An infinite sequence is said to be convergent with limit L if

$$\lim_{n\to\infty} a_n = L.$$

If no limit (finite) limit exists, the sequence is said to be divergent.

Theorem (on continuous functions)

Theorem: If $\lim_{n\to\infty} a_n = L$ and f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Example: Determine the limit

$$\lim_{n\to\infty} \exp\left(\frac{1}{n^2}\right)$$

$$= \exp\left(\lim_{n\to\infty} \frac{1}{n^2}\right)$$

$$= e^0$$

$$= 1$$

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

$f(x) = e^x$

is continuous

@ zero

A Special Sequence

Let r be a real number. Determine the convergence or divergence of the sequence

$$a_n = r^n.$$

Case 1: $r = 1$ $\quad a_n = 1^n = 1 \quad \lim_{n\to\infty} 1 = 1$

convergent w/ limit 1

Case 2: $r = -1$ $\quad a_n = (-1)^n \quad \lim_{n\to\infty} (-1)^n$ DNE

divergent

Case 3: $|r| < 1$ $\quad \lim_{n\to\infty} a_n = 0$

note $|r|^n \to 0$ as successive powers

get smaller.

The sequence converges w/ limit 0.

Case 4: $|r| > 1$ $\qquad \lim_{n\to\infty} a_n$ DNE

The sequence is divergent.

$$a_n = r^n \quad \begin{cases} \text{Convergent to } 1 & r = 1 \\ \text{Divergent} & r = -1 \\ \text{Divergent} & |r| > 1 \\ \text{Convergent to } 0 \text{ if } & |r| < 1 \end{cases}$$

Monotone Sequences

Definition: A sequence is **increasing** (or strictly increasing) if $a_n < a_{n+1}$ for all n. That is, an increasing sequence would satisfy

$$a_1 < a_2 < \cdots < a_n < \cdots .$$

A sequence is **decreasing** (or strictly decreasing) if $a_n > a_{n+1}$ for all n. That is, a decreasing sequence would satisfy

$$a_1 > a_2 > \cdots > a_n > \cdots .$$

A sequence that is either increasing or is decreasing is called **monotonic.**

For example,

$a_n = \dfrac{1}{n}$ is decreasing, $\quad b_n = (-1)^n$ oscillates, and

$c_n = 2^n$ is increasing.

Example Using a function to determine if a sequence is monotone:

Let $a_n = \dfrac{n}{n^2+1}$. Show that a_n is a decreasing sequence.

Let $f(x) = \dfrac{x}{x^2+1}$ note $f(n) = a_n$ for integers n

Take $f'(x)$

$$f'(x) = \frac{x^2+1 - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

for $x=1$, $f'(1) = 0$

for $x > 1$, $1 - x^2 < 0$ so $f'(x) < 0$

$a_1 = \frac{1}{1+1} = \frac{1}{2}$, $a_2 = \frac{2}{5}$

so $a_2 < a_1$

for $n \geq 2$ $f(n+1) < f(n)$

since f is decreasing.

i.e. $a_{n+1} < a_n$ so the

sequence is decreasing.

Boundedness

Definition: A sequence $\{a_n\}$ is **bounded above** if there exists a number M such that

$$a_n \leq M \quad \text{for all} \quad n \geq 1.$$

A sequence $\{a_n\}$ is **bounded below** if there exists a number m such that

$$a_n \geq m \quad \text{for all} \quad n \geq 1.$$

A sequence that is both bounded above and bounded below is called a **bounded sequence.**

Example

Determine if the sequence is bounded above, bounded below, and/or is a bounded sequence.

(a) $a_n = 2^n$ $\{1, 2, 4, 8, 16, \ldots\}$

$2^n \geq 1$ for $n \geq 0$ bounded below

$2^n \to \infty$ so it's not bounded above.

(b) $b_n = 1 + (-1)^n$ $\{2, 0, 2, 0, 2, 0, \ldots\}$

$b_n \geq 0$ for all n, it's bounded below

$b_n \leq 2$ for all n, it's bounded above,

It is a bounded sequence.

Example continued...

Determine if the sequence is bounded above, bounded below, and/or is a bounded sequence.

(c) $c_n = \begin{cases} \frac{3}{n+2}, & n \text{ is even} \\ -4n, & n \text{ is odd} \end{cases}$ (assume $n \geq 0$)

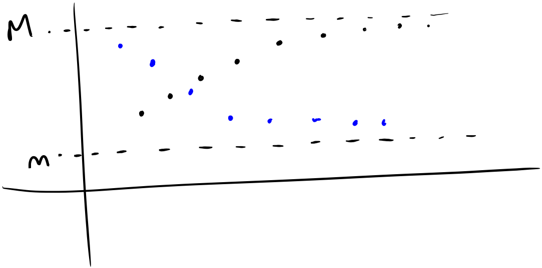
$$\left\{ \frac{3}{2}, -4, \frac{3}{4}, -12, \frac{3}{6}, -20, \frac{3}{8}, -28, \dots \right\}$$

The odd terms tend to $-\infty$. It's not bounded below.

It is bounded above since $c_n \leq \frac{3}{2}$ for all $n \geq 0$.

The Monotonic Sequence Theorem

Theorem: Every bounded monotonic sequence is convergent.



- increasing
- decreasing

Example: Consider the sequence given by

$$a_1 = \sqrt{2}, \quad a_2 = \sqrt{2\sqrt{2}}, \quad a_3 = \sqrt{2\sqrt{2\sqrt{2}}}, \quad \cdots \quad a_n = \sqrt{2a_{n-1}}.$$

It can be shown that

(1) a_n is strictly increasing, and (2) that $1 \le a_n \le 3$ for every n.

it's monotonic — it's bounded

Discuss the convergence or divergence of $\{a_n\}$. If convergent, find its limit.

a_n is monotonic and bounded, hence it is convergent.

Since it is convergent $\lim_{n\to\infty} a_n = L$ for some finite number L.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{2a_{n-1}} = \sqrt{2\lim_{n\to\infty} a_{n-1}}$$

$$L = \sqrt{2L} \implies L^2 = 2L$$

$$L^2 - 2L = 0 \implies L(L-2) = 0$$

$\cancel{L=0}$ or $L=2$

extraneous

The limit is 2.

Section 11.2: Series

Definition: Suppose we have an infinite sequence of numbers $\{a_1, a_2, \ldots\}$. We can consider summing them to form the expression

$$a_1 + a_2 + \cdots + a_n + \cdots$$

Such an expression is called a **series**. We may call it an **infinite series** to highlight that there are infinitely many summands.

Notation: We'll denote sums using a capital sigma (Greek letter "S") as follows:

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k.$$

If the limits, starting from $k = 1$ and going to ∞, are understood, we may simply write $\sum a_k$.

Sigma Notation

maximum index value

$$\sum_{k=1}^{\infty} a_k$$

summands the k^{th} term is a_k

index

first index value

Examples:

Some series would obviously give rise to a sum that is an infinty–e.g. the series

$$1 + 2 + 3 + \cdots + n + \cdots$$

Others give a well defined, finite sum inspite of there being infinitely many term. For example, it can be shown that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = 1.$$

Partial Sums

Definition: Let $\sum a_k$ be a series. The **sequence of partial sums** is the sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k. \end{aligned}$$

Example: For the series $\sum_{k=1}^{\infty} \frac{1}{2^k}$, find the first three terms in the sequence of partial sums, s_1, s_2, and s_3.

$$s_1 = \frac{1}{2^1} = \frac{1}{2}$$

$$S_2 = \frac{1}{2^1} + \frac{1}{2^2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

Note : in general $S_{n+1} = S_n + a_{n+1}$