March 16 Math 3260 sec. 55 Spring 2020

Section 4.4: Coordinate Systems

Definition: (Coordinate Vectors) Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an **ordered** basis of the vector space V. For each $\mathbf{x}$ in V we define the **coordinate vector of x relative to the basis** $\mathcal{B}$ to be the unique vector $(c_1, \ldots, c_n)$ in $\mathbb{R}^n$ where these entries are the weights $\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n$.

We'll use the notation

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = [\mathbf{x}]_{\mathcal{B}}.$$

Coordinates in $\mathbb{R}^n$

Change of Coordinates in $\mathbb{R}^n$: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis of $\mathbb{R}^n$. Then the change of coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the linear transformation defined by

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$$

where the matrix

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n].$$

Theorem: Coordinate Mapping

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a **one to one** mapping of V **onto** $\mathbb{R}^n$.

Remark: When such a mapping exists, we say that V is **isomorphic** to $\mathbb{R}^n$. Properties of subsets of V, such as linear dependence, can be discerned from the coordinate vectors in $\mathbb{R}^n$.

Example

Use coordinate vectors to determine if the set $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ is linearly dependent or independent in $\mathbb{P}_2$.

$$\mathbf{p}(t) = 1 - 2t^2, \quad \mathbf{q}(t) = 3t + t^2, \quad \mathbf{r}(t) = 1 + t$$

Section 4.5: Dimension of a Vector Space

Theorem: If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set of vectors in V containing *more than n vectors* is linearly dependent.

We saw an example of this before: a set of p vectors in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Why is the set $\{1+t, 1-2t, 2+4t\}$ linearly dependent in $\mathbb{P}_1$?

Dimension

Corollary: If vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then every basis of V consist of exactly n vectors.

- What this says is that the number of basis elements for a given vector space is fixed.
- This number can be used, unambiguously, as a characteristic of the vector space. This leads to the definition of **dimension**.

Dimension of a Vector Space

Definition: If V is spanned by a finite set, then V is called **finite dimensional.**

In this case, the dimension of V

$$\dim V \ = \ \text{the number of vectors in any basis of } V.$$

The dimension of the vector space $\{\mathbf{0}\}$ containing only the zero vector is defined to be zero—i.e.

$$\dim\{\mathbf{0}\} = 0.$$

If V is not spanned by a finite set[1], then V is said to be **infinite dimensional.**

[1] $C^0(\mathbb{R})$ is an example of an infinite dimensional vector space.

Examples

(a) Find $\dim(\mathbb{R}^n)$.

Examples

(b) Determine dim Col A where $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$.

Some Geometry in $\mathbb{R}^3$

The subspaces of $\mathbb{R}^3$ of various dimensions are:

- **Zero:** One point, the origin.
- **One:** Any line through the origin, e.g. $\mathsf{Span}\{\mathbf{u}\}$.
- **Two:** Any plane that contains the origin, e.g. $\mathsf{Span}\{\mathbf{u}, \mathbf{v}\}$.
- **Three:** All of $\mathbb{R}^3$

Note: It is assumed that vectors **u** and **v** are linearly independent, nonzero vectors.

Subspaces and Dimension

Theorem: Let H be a subspace of a finite dimensional vector space V. Then H is finite dimensional and

$$\dim H \leq \dim V.$$

Moreover, any linearly independent subset of H can be expanded if needed to form a basis for H.

We already knew that if we had a spanning set, we could obtain a basis (by getting rid of duplicating vectors).

This says if we start with a linearly independent set, we can add linearly independent vectors as needed , until we get a spanning set.

Subspaces and Dimension

Theorem: Let V be a vector space with $\dim V = p$ where $p \geq 1$. Any linearly independent set in V containing exactly p vectors is a basis for V. Similarly, any spanning set consisting of exactly p vectors in V is necessarily a basis for V.

That is: If a set of p vectors in a p-dimensional vectors space V

(1) spans V, it is **automatically** linearly independent.

(2) is linearly independent, it **automatically** spans V.

Column and Null Spaces

Theorem: Let A be an $m \times n$ matrix. Then

$$\dim \text{Nul}A \;=\; \text{the number of free variables in the equation } \; A\mathbf{x} = \mathbf{0},$$

and

$$\dim \text{Col}A \;=\; \text{the number of pivot positions in } \; A.$$

Example

Find the dimensions of the null and columns spaces of the matrix A.

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & 1 & -7 & -1 \\ 3 & 0 & 6 & 1 \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$