Mar. 17 Math 2254H sec 015H Spring 2015

Section 11.2: Series

Definition: Suppose we have an infinite sequence of numbers $\{a_1, a_2, \ldots\}$. We can consider summing them to form the expression

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k$$

Such an expression is called a **series**. We may call it an **infinite series** to highlight that there are infinitely many summands.

Definition: Let $\sum a_k$ be a series. The **sequence of partial sums** is the sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k. \end{aligned}$$

Convergence or Divergence

Definition: Given a series $\sum a_k$, let $\{s_n\}$ denote the sequence of partial sums. If the sequence $\{s_n\}$ converges with limit s, that is

$$\text{if} \quad \lim_{n\to\infty} s_n = s,$$

then the series $\sum a_k$ is said to be **convergent**, and s is called the **sum** of the series. In this case, we write

$$\sum_{k=1}^{\infty} a_k = s.$$

If the sequence $\{s_n\}$ is divergent, then the series is said to be **divergent**.

Remark: A convergence or divergence of a series is defined in terms of the convergence or divergence of its sequence of partial sums.

Remark: If a sequence $\sum a_k$ converges, it is a **number**.

Example

Show that the series converges and find its sum.

$$\sum_{k=1}^{\infty} \frac{1}{k^2+k}$$

$S_1 = \frac{1}{1^2+1} = \frac{1}{2}$

$S_2 = \frac{1}{1^2+1} + \frac{1}{2^2+2} = \frac{1}{2} + \frac{1}{6}$

$S_3 = \frac{1}{1^2+1} + \frac{1}{2^2+2} + \frac{1}{3^2+3} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12}$

$\vdots$

Partial fraction

$$\frac{1}{k^2+k} = \frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1} \qquad k(k+1)$$

$$1 = A(k+1) + Bk = (A+B)k + A$$

$$A+B=0 \implies B=-A \quad \text{and} \quad A=1$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2+k} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

Sequence of partial sums

$$S_1 = 1 - \frac{1}{2}$$

$$S_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}$$

$$\vdots$$

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$

The series converges and

$$\sum_{k=1}^{\infty} \frac{1}{k^2+k} = 1.$$

A Divergent Series

Use the well known result $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ to investigate the convergence or divergence of the series

$$\sum_{k=1}^{\infty} k.$$

$S_1 = 1$

$S_2 = 1 + 2 = 3$

$S_3 = 1 + 2 + 3 = 6$

$\vdots$

$S_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

$\lim_{n\to\infty} S_n =$

$\lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$

$\sum_{k=1}^{\infty} k$ is divergent.

Geometric Series

The series

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots = \sum_{n=0}^{\infty} ar^n, \quad a \neq 0^1$$

is called **geometric series.** The number r is called the **common ratio.**

Investigate the convergence or divergence of this series.

$$S_0 = a, \quad S_1 = a + ar, \quad S_2 = a + ar + ar^2$$

$$\vdots$$

$$S_N = a + ar + ar^2 + \ldots + ar^N$$

[1] Many authors, including Stewart, prefer to have the index start at 1 instead of zero, and to replace the power n with the power $n-1$–i.e. they write $\sum_{n=1}^{\infty} ar^{n-1}$.

Case 1: $r=1$ the series is

$$\sum_{n=0}^{\infty} a \quad \text{and} \quad s_N = a(N+1)$$

$$\lim_{N\to\infty} s_N = \lim_{N\to\infty} a(N+1) \quad \text{DNE}$$

The series diverges.

Case 2: $r \neq 1$

$$s_N = a + ar + \dots + ar^N$$

$$rs_N = ar + ar^2 + \dots + ar^N + ar^{N+1}$$

$$s_N - rs_N = a - ar^{N+1} \implies$$

$$(1-r)S_N = a(1-r^{N+1})$$

$$\Rightarrow \quad S_N = \frac{a(1-r^{N+1})}{1-r}$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a(1-r^{N+1})}{1-r}$$

$$= \begin{cases} \frac{a}{1-r} & \text{if } |r|<1 \\ \text{DNE} & \text{if } |r| \geq 1 \end{cases}$$

$$\boxed{\begin{array}{l} \text{If } |r|<1 \\ \lim_{n\to\infty} r^n = 0 \\ \text{If } |r| \geq 1 \\ \lim_{n\to\infty} r^n \text{ DNE} \end{array}}$$

Geometric Series

Theorem: The series $a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n$ is convergent if $|r| < 1$. In this case,

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad |r| < 1.$$

If $|r| \geq 1$, the series is divergent.

Examples:

Determine the convergence or divergence of the series. If convergent, find the sum.

If it's geometric, then

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{3n+1}}{5^{n-1}}$

$\dfrac{a_{n+1}}{a_n}$ is constant

$$2^{3n+1} = 2 \cdot 2^{3n} = 2 \cdot (2^3)^n = 2 \cdot 8^n$$

$$5^{n-1} = \frac{1}{5} \cdot 5^n \qquad \frac{2^{3n+1}}{5^{n-1}} = \frac{2 \cdot 8^n}{\frac{1}{5} 5^n} = 10\left(\frac{8}{5}\right)^n$$

Our series is $\displaystyle\sum_{n=0}^{\infty} 10\left(\frac{8}{5}\right)^n$ $\quad r = \frac{8}{5} > 1$ $\quad \boxed{\text{divergent}}$

Examples continued

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{5^{n+1}}{3^{2n-1}}$

$5^{n+1} = 5 \cdot 5^n$

$3^{2n-1} = \frac{1}{3} \cdot 3^{2n} = \frac{1}{3} \cdot 9^n$

$$= \sum_{n=0}^{\infty} \frac{5 \cdot 5^n}{\frac{1}{3} \cdot 9^n}$$

Convergent $|r| = \left|\frac{5}{9}\right| < 1$

$$= \sum_{n=0}^{\infty} 15 \left(\frac{5}{9}\right)^n$$

$$= \frac{15}{1 - 5/9} \cdot \frac{9}{9} = \frac{135}{9-5} = \frac{135}{4}$$

Examples...last one

(c) $\displaystyle\sum_{n=2}^{\infty} 5\left(\frac{1}{4}\right)^n = \sum_{k=0}^{\infty} 5\left(\frac{1}{4}\right)^{k+2}$

$|r| = \left|\frac{1}{4}\right| < 1$

$= \displaystyle\sum_{k=0}^{\infty} 5\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right)^k$

$= \displaystyle\sum_{k=0}^{\infty} \frac{5}{16}\left(\frac{1}{4}\right)^k = \frac{5/16}{1-\frac{1}{4}} \cdot \frac{16}{16}$

$= \dfrac{5}{16-4} = \dfrac{5}{12}$

Telescoping Sum

The series $\sum \frac{1}{k(k+1)}$ is an example of a *telescoping* series.

Definition: A series of the form

$$\sum_{k=1}^{\infty} (a_k - a_{k+1})$$

is called a **telescoping series.** The sequence of partial sums is determined to be

$$s_n = a_1 - a_{n+1}$$

and is convergent if and only if $\lim_{n\to\infty} a_n$ exists (as a finite number).

A Special Series: The Harmonic Series

Definition: The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

is called the **harmonic series.**

Theorem: The harmonic series is divergent.

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2}$$

$$\vdots$$

$$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$\vdots$$

$$S_{2^n} > 1 + \frac{n}{2}$$

$$\lim_{n\to\infty}\left(1+\frac{n}{2}\right) = \infty$$

So $$\lim_{n\to\infty} S_{2^n} = \infty$$

Hence $\{S_n\}$ diverges and

$$\sum_{n=1}^{\infty}\frac{1}{n}$$ diverges.