March 17 Math 2306 sec 58 Spring 2016

Section 10: Variation of Parameters

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x) = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

We made the assumption that

$$u_1' y_1 + u_2' y_2 = 0$$

With this assumption, we found the first and second derivatives of y_p to be

$$y_p' = u_1 y_1' + u_2 y_2', \quad \text{and}$$
$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''.$$

We need to substitute this into the ODE.

Remember that $\quad y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

we need $$y_p'' + P(x)y_p' + Q(x)y_p = g(x)$$

$$u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'' + P(x)\left(u_1y_1' + u_2y_2'\right) + Q(x)\left(u_1y_1 + u_2y_2\right) = g(x)$$

$$u_1'y_1' + u_2'y_2' + u_1\Big(\underbrace{y_1'' + P(x)y_1' + Q(x)y_1}_{=0}\Big) + u_2\Big(\underbrace{y_2'' + P(x)y_2' + Q(x)y_2}_{=0}\Big) = g(x)$$

as y_1, y_2 solve the homogeneous equation.

$$\Rightarrow \quad u_1'y_1' + u_2'y_2' = g(x)$$

We have the system

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = g(x)$$

In matrix form

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

Let $W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = 0 - g y_2 = -g y_2$

$W_2 \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = y_1 g - 0 = y_1 g$

Then $u_1' = \dfrac{W_1}{W}$ and $u_2' = \dfrac{W_2}{W}$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$

Finally, $u_1 = \displaystyle\int \frac{-y_2 g}{W}\,dx$ and

$$u_2 = \int \frac{y_1 g}{W}\,dx$$

Example:

Solve the ODE $y'' + y = \tan x$.

Find y_c: $\quad y'' + y = 0$, $\quad m^2 + 1 = 0 \implies m = \pm i$ $\qquad \alpha = 0$, $\beta = 1$

$y_1 = e^{0x}\cos x = \cos x$

$y_2 = e^{0x}\sin x = \sin x$

Wronskian $\quad W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

The eqn is in standard form, so $g(x) = \tan x$

$$u_1 = \int \frac{-y_2 g}{W}\,dx = \int \frac{-\sin x \tan x}{1}\,dx = -\int \sin x \tan x\,dx$$

$$= -\int \sin x \left(\frac{\sin x}{\cos x}\right) dx = -\int \frac{\sin^2 x}{\cos x}\,dx = -\int \frac{1-\cos^2 x}{\cos x}\,dx$$

$$= -\int (\sec x - \cos x)\,dx = -\ln|\sec x + \tan x| + \sin x$$

$$u_2 = \int \frac{y_1 g}{W}\,dx = \int \frac{\cos x \tan x}{1}\,dx = \int \cos x \left(\frac{\sin x}{\cos x}\right) dx$$

$$= \int \sin x\,dx = -\cos x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(-\ln|\sec x + \tan x| + \sin x\right)\cos x + (-\cos x)\sin x$$

$$= -\cos x \ln|\sec x + \tan x| + \sin x \cos x - \cos x \sin x$$

$$\boxed{y_p = -\cos x \ln|\sec x + \tan x|}$$

The general solution is

$$\boxed{y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|}$$

Example:

Solve the ODE

$$y'' - 2y' + y = \frac{e^x}{1+x^2}.$$

Find y_c: $m^2 - 2m + 1 = 0$ $(m-1)^2 = 0$ $\Rightarrow$ $m = 1$ repeated

$y_1 = e^x$, $y_2 = xe^x$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^x(e^x + xe^x) - e^x xe^x$$

$$= e^{2x} + xe^{2x} - xe^{2x}$$

$$= e^{2x}$$

Eqn is in standard form so $g(x) = \frac{e^x}{1+x^2}$

$$u_1 = \int \frac{-y_2 g}{W}\,dx = -\int \frac{xe^x\left(\frac{e^x}{1+x^2}\right)}{e^{2x}}\,dx = -\int \frac{1}{e^{2x}}\,\frac{xe^{2x}}{1+x^2}\,dx$$

$$= -\int \frac{x}{1+x^2}\,dx = -\frac{1}{2}\ln(1+x^2)$$

Take $u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$u_2 = \int \frac{y_1 g}{W}\,dx = \int \frac{e^x\left(\frac{e^x}{1+x^2}\right)}{e^{2x}}\,dx$$

$$= \int \frac{1}{e^{2x}}\,\frac{e^{2x}}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,dx = \tan^{-1}x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{-1}{2}\ln(1+x^2)\,e^x + \tan^{-1}x\,(xe^x)$$

$$= -\frac{1}{2}e^x\ln(1+x^2) + xe^x\tan^{-1}x$$

The general solution is

$$y = c_1 e^x + c_2 x e^x - \frac{1}{2}e^x\ln(1+x^2) + xe^x\tan^{-1}x$$