March 17 Math 2306 sec 59 Spring 2016

Section 10: Variation of Parameters

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

We found that the functions u_1 and u_2 were determined as

$$u_1(x) = \int \frac{-y_2(x)g(x)}{W}\,dx \quad \text{and} \quad u_2(x) = \int \frac{y_1(x)g(x)}{W}\,dx$$

where W is the Wronskian of y_1 and y_2.

Example:

Solve the ODE $y'' + y = \tan x$.

Find y_1, y_2 : $y'' + y = 0$ $m^2 + 1 = 0$ $m^2 = -1$, $m = \pm i$

$m = \alpha \pm i\beta$ $\alpha = 0$, $\beta = 1$

$y_1 = e^{0x} \cos x = \cos x$

$y_2 = e^{0x} \sin x = \sin x$

Wronskian

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

The equation is in standard form, so

$g(x) = \tan x$

$$u_1 = \int \frac{-y_2 g}{W}\,dx = \int -\frac{\sin x \tan x}{1}\,dx = -\int \sin x \tan x\,dx$$

$$= -\int \sin x\left(\frac{\sin x}{\cos x}\right)dx = -\int \frac{\sin^2 x}{\cos x}\,dx = -\int \frac{1-\cos^2 x}{\cos x}\,dx$$

$$= -\int (\sec x - \cos x)\,dx$$

$$= -\ln|\sec x + \tan x| + \sin x$$

$$u_2 = \int \frac{y_1 g}{W}\,dx = \int \frac{\cos x \tan x}{1}\,dx$$

$$= \int \cos x \left(\frac{\sin x}{\cos x}\right) dx = \int \sin x\,dx = -\cos x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(-\ln|\sec x + \tan x| + \sin x\right)\cos x + \left(-\cos x\right)\sin x$$

$$= -\cos x \ln|\sec x + \tan x| + \sin x \cos x - \cos x \sin x$$

$$\Longrightarrow \quad y_p = -\cos x \ln|\sec x + \tan x|$$

The general solution is

$$y = c_1 \cos x + c_2 \sin x - \cos x \ln\left|\sec x + \tan x\right|$$

Example:

Solve the ODE

$$y'' - 2y' + y = \frac{e^x}{1+x^2}.$$

Find y_1, y_2 : $m^2 - 2m + 1 = 0$ $(m-1)^2 = 0 \Rightarrow m = 1$ repeated.

$y_1 = e^x$, $y_2 = xe^x$

Wronskian

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^x(xe^x + e^x) - e^x(xe^x)$$

$$= xe^{2x} + e^{2x} - xe^{2x}$$

$$= e^{2x}$$

The eqn. is in standard form

so $g(x) = \dfrac{e^x}{1+x^2}$

$$u_1 = \int \frac{-y_2 g}{W}\,dx = \int \frac{-xe^x\left(\frac{e^x}{1+x^2}\right)}{e^{2x}}\,dx = \int \frac{-1}{e^{2x}}\,\frac{xe^{2x}}{1+x^2}\,dx$$

$$= -\int \frac{x}{1+x^2}\,dx$$

$u = 1+x^2$, $du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$$= \frac{-1}{2}\int \frac{du}{u} = \frac{-1}{2}\ln|u| = \frac{-1}{2}\ln|1+x^2|$$

$$u_2 = \int \frac{y_1 g}{W}\,dx = \int \frac{e^x\left(\frac{e^x}{1+x^2}\right)}{e^{2x}}\,dx = \int \frac{1}{e^{2x}}\,\frac{e^{2x}}{1+x^2}\,dx$$

$$= \int \frac{1}{1+x^2}\,dx = \tan^{-1}x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= -\frac{1}{2} \ln|1+x^2|\, e^x + \tan^{-1}x \left(x e^x\right)$$

$$y_p = \frac{-e^x}{2} \ln|1+x^2| + x e^x \tan^{-1}x$$

The general solution is

$$y = c_1 e^x + c_2 x e^x - \frac{e^x}{2} \ln|1+x^2| + x e^x \tan^{-1}x \,.$$

Example:

Solve the ODE
$$x^2y'' + xy' - 4y = \ln x,$$
given that $y_c = c_1x^2 + c_2x^{-2}$ is the complementary solution.

We're given $y_1 = x^2$, $y_2 = x^{-2}$

Wronskian $W = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = x^2(-2x^{-3}) - 2x(x^{-2})$

$= -2x^{-1} - 2x^{-1} = -4x^{-1}$

Standard form: $y'' + \frac{1}{x}y' - \frac{4}{x^2}y = \frac{\ln x}{x^2}$

$g(x) = \frac{\ln x}{x^2}$

$$u_1 = \int \frac{-y_2 g}{W}\,dx = \int \frac{-x^{-2}\left(\frac{\ln x}{x^2}\right)}{-4x^{-1}}\,dx = \frac{1}{4}\int x^{-2}\cdot x\cdot x^{-2}\ln x\ dx$$

$$= \frac{1}{4}\int x^{-3}\ln x\ dx$$

$u = \ln x$ $\qquad du = \frac{1}{x}\,dx$

$v = \frac{x^{-2}}{-2}$ $\qquad dv = x^{-3}\,dx$

$$= \frac{1}{4}\left[\frac{-1}{2}x^{-2}\ln x - \int \frac{-1}{2}x^{-2}\cdot x^{-1}\,dx\right]$$

$$= \frac{1}{4}\left[\frac{-1}{2}x^{-2}\ln x + \frac{1}{2}\cdot\frac{x^{-2}}{-2}\right] = \frac{-1}{8}x^{-2}\ln x - \frac{1}{16}x^{-2}$$

$$u_2 = \int \frac{y_1 g}{W}\,dx = \int \frac{x^2\left(\frac{\ln x}{x^2}\right)}{-4x^{-1}}\,dx = \frac{-1}{4}\int x\ln x\,dx$$

$$u = \ln x \qquad du = \frac{1}{x}\,dx$$

$$v = \frac{x^2}{2} \qquad dv = x\,dx$$

$$= \frac{-1}{4}\left[\frac{x^2}{2}\ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\,dx\right]$$

$$= \frac{-1}{4}\left[\frac{x^2}{2}\ln x - \frac{1}{4}x^2\right] = \frac{-1}{8}x^2\ln x + \frac{1}{16}x^2$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p = \left(\frac{-1}{8}x^{-2}\ln x - \frac{1}{16}x^{-2}\right)x^2 + \left(\frac{-1}{8}x^2\ln x + \frac{1}{16}x^2\right)x^{-2}$$

$$y_p = \frac{-1}{8}\ln x - \frac{1}{16} - \frac{1}{8}\ln x + \frac{1}{16}$$

$$y_p = \frac{-1}{4}\ln x$$

The general solution is

$$\boxed{y = C_1 x^2 + C_2 x^{-2} - \frac{1}{4}\ln x.}$$