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Section 4.3: Interpolating Using Spline Functions

Suppose that the n data points (x_j, y_j), $j = 1, 2, \ldots, n$ are given. Assume $x_1 = a$, $x_{j-1} < x_j$ and $x_n = b$. There exists a function $s(x)$ that interpolates these points—i.e.

$$s(x_j) = y_j, \quad \text{for} \quad j = 1, \ldots, n$$

satisfying the following properties:

S1. $s(x)$ is a polynomial of degree ≤ 3 on each interval $[x_j, x_{j+1}]$, $j = 1, \ldots n-1$.

S2. $s(x)$, $s'(x)$, and $s''(x)$ are continuous on $[a, b]$.

S3. $s''(x_1) = 0$ and $s''(x_n) = 0$

The curve $s(x)$ is called the *natural cubic spline* that interpolates the data.

Constructing the Natural Cubic Spline

We start with data in order of increasing x $\{(x_j, y_j) \,|\, j = 1, \ldots, n\}$. Define the distances between adjacent points

$$h_j = x_{j+1} - x_j$$

Then we introduce the constants $M_1, \ldots, M_n$ where

$$M_1 = M_n = 0, \quad \text{and}$$

$$\frac{h_{j-1}}{6} M_{j-1} + \frac{h_j + h_{j-1}}{3} M_j + \frac{h_j}{6} M_{j+1} = \frac{y_{j+1} - y_j}{h_j} - \frac{y_j - y_{j-1}}{h_{j-1}}$$

$$\text{for} \quad j = 2, \ldots, n-1.$$

Constructing the Natural Cubic Spline

Finally we build the cubic spline function $s(x)$ on each subinterval. On the interval $[x_j, x_{j+1}]$

$$s(x) = \frac{M_j}{6h_j}(x_{j+1} - x)^3 + \frac{M_{j+1}}{6h_j}(x - x_j)^3 + \frac{y_j}{h_j}(x_{j+1} - x) + \frac{y_{j+1}}{h_j}(x - x_j) -$$

$$-\frac{h_j}{6}\left[M_j(x_{j+1} - x) + M_{j+1}(x - x_j)\right], \quad x_j \le x \le x_{j+1}$$

$$j = 1, \ldots, n-1$$

Cubic Spline Example 2

Find the natural cubic spline interpolating the data

$$\{(1,12),(2,6),(3,4),(4,3)\}$$

Here $h_j = h = 1$ for all $j = 1, \ldots, 3$

We need M_1, M_2, M_3, M_4.

$M_1 = M_4 = 0$, so we really only need M_2, M_3.

$j=2$ $\quad \underset{\overset{\shortparallel}{0}}{M_1} + 4M_2 + M_3 = 6(y_3 - 2y_2 + y_1)$

$$4M_2 + M_3 = 6(4 - 2\cdot 6 + 12) = 24$$

$j = 3$ $$M_2 + 4M_3 + \underset{\overset{\shortparallel}{0}}{M_4} = 6(y_4 - 2y_3 + y_2)$$

$$M_2 + 4M_3 = 6(3 - 2\cdot 4 + 6) = 6$$

we need to solve

$$4M_2 + M_3 = 24$$
$$M_2 + 4M_3 = 6$$

Take -4 times eqn. 2 and add

$$\begin{array}{r} 4M_2 + M_3 = 24 \\ -4M_2 - 16M_3 = -24 \\ \hline -15M_3 = 0 \end{array} \implies M_3 = 0$$

$$4M_2 = 24 \implies M_2 = 6$$

So $M_1 = 0, \ M_2 = 6, \ M_3 = 0, \ M_4 = 0$

on $[1,2]$, $j=1$

$$S(x) = \frac{M_1^{=0}}{6h}(x_2-x)^3 + \frac{M_2}{6h}(x-x_1)^3 + \frac{y_1}{h}(x_2-x) + \frac{y_2}{h}(x-x_1)$$

$$-\frac{h}{6}\left[M_1^{=0}(x_2-x) + M_2(x-x_1)\right]$$

$$= \frac{6}{6}(x-1)^3 + \frac{12}{1}(2-x) + \frac{6}{1}(x-1) - \frac{1}{6}\left[6(x-1)\right]$$

$$= (x-1)^3 + 12(2-x) + 6(x-1) - (x-1)$$

$$= x^3 - 3x^2 + 3x - 1 + 24 - 12x + 6x - 6 - x + 1$$

$$= x^3 - 3x^2 - 4x + 18$$

on $[2,3]$ $\quad j=2$

$$S(x) = \frac{M_2}{6h}(x_3-x)^3 + \frac{\overset{=0}{M_3}}{6h}(x-x_2)^3 + \frac{y_2}{h}(x_3-x) + \frac{y_3}{h}(x-x_2)$$

$$-\frac{h}{6}\left[M_2(x_3-x) + \overset{=0}{M_3}(x-x_2)\right]$$

$$= \frac{6}{6}(3-x)^3 + \frac{6}{1}(3-x) + \frac{4}{1}(x-2) - \frac{1}{6}\left[6(3-x)\right]$$

$$= (3-x)^3 + 6(3-x) + 4(x-2) - (3-x)$$

$$= 27 - 27x + 9x^2 - x^3 + 18 - 6x + 4x - 8 - 3 + x$$

$$= -x^3 + 9x^2 - 28x + 34$$

on $[3,4]$ $\quad j=3$

$$S(x) = \frac{\overset{=0}{M_3}}{6h}(x_4 - x) + \frac{\overset{=0}{M_4}}{6h}(x - x_3) + \frac{y_3}{h}(x_4 - x) + \frac{y_4}{h}(x - x_3)$$

$$- \frac{h}{6}\left[\overset{=0}{M_3}(x_4 - x) + \overset{=0}{M_4}(x - x_3)\right]$$

$$= \frac{4}{1}(4 - x) + \frac{3}{1}(x - 3)$$

$$= 16 - 4x + 3x - 9$$

$$= -x + 7$$

Example 2 Results

Find the natural cubic spline interpolating the data
$\{(1,12),(2,6),(3,4),(4,3)\}$.

We determined the values M_j to be

$$M_1 = 0, \quad M_2 = 6, \quad M_3 = 0, \quad M_4 = 0.$$

And we found the natural cubic spline function

$$s(x) = \begin{cases} x^3 - 3x^2 - 4x + 18, & 1 \le x \le 2 \\ -x^3 + 9x^2 - 28x + 34, & 2 \le x \le 3 \\ -x + 7, & 3 \le x \le 4 \end{cases}$$

Section 5.1: Numerical Integration, the Trapezoid and Simpson Rules

Our goal is to evaluate a definite integral

$$I(f) = \int_a^b f(x)\,dx$$

We may recall the Fundamental Theorem of Calculus tells us

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

provided $F(x)$ is any anti-derivative of $f(x)$[1].

[1] Assuming f has an anti-derivative.

Numerical Integration

Even a rather tame function may not have an anti-derivative that can be written in terms of elementary functions. Or an anti-derivative may be so complicated as to make integration exceedingly difficult. For example:

$$\int_0^1 e^{x^2}\,dx \quad \text{no elementary anti-der. exists}$$

$$\int_0^1 \frac{dx}{x^5+1} \quad \text{the anti-derivative is very complicated!}$$

▸ anti-derivative of $1/(x^5+1)$

Numerical Integration

One Approach: Approximate f with some simpler function, and integrate that.

We have two choices for the approximation: (1) a Taylor polynomial, or (2) an interpolating polynomial.

The first rule involves Linear Interpolation. This is called the **Trapezoid** rule.

Trapezoid Rule

$$\int_a^b f(x)\,dx \approx \int_a^b P_1(x)\,dx$$

$$\text{where} \quad P_1(x) = \frac{(b-x)f(a) + (x-a)f(b)}{b-a},$$

the linear interpolation of $f(x)$ on $[a, b]$.

$$\text{Note:} \quad P_1(x) = \frac{f(a)}{b-a}(b-x) + \frac{f(b)}{b-a}(x-a).$$

Trapezoid Rule

Show that

$$\int_a^b \frac{f(a)}{b-a}(b-x)\,dx = \frac{1}{2}(b-a)f(a).$$

$$\frac{f(a)}{b-a}\int_a^b (b-x)\,dx = \frac{f(a)}{b-a}\left[bx - \frac{x^2}{2}\right|_a^b$$

$$= \frac{f(a)}{b-a}\left[b\cdot b - \frac{b^2}{2} - \left(b\cdot a - \frac{a^2}{2}\right)\right]$$

$$= \frac{f(a)}{b-a}\left[b^2 - \frac{1}{2}b^2 - ab + \frac{a^2}{2}\right]$$

Continued...[2]

$$= \frac{f(a)}{b-a}\left[\frac{1}{2}b^2 - ab + \frac{1}{2}a^2\right]$$

$$= \frac{f(a)}{b-a}\,\frac{1}{2}\left(b^2 - 2ab + a^2\right)$$

$$= \frac{1}{2}\,\frac{f(a)}{b-a}(b-a)^2 \qquad = \frac{1}{2}f(a)(b-a)$$

$$= \frac{1}{2}(b-a)f(a)$$

[2] A similar computation shows that

$$\int_a^b \frac{f(b)}{b-a}(x-a)\,dx = \frac{1}{2}(b-a)f(b).$$

Trapezoid Rule

$$\int_a^b P_1(x)\,dx = \frac{1}{2}(b-a)\left[f(b)+f(a)\right]$$

We'll call the right side $T_1(f)$, and we can write

$$\int_a^b f(x)\,dx \approx T_1(f).$$

The trapezoid rule with one interval is given by

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)\left[f(b)+f(a)\right] = T_1(f).$$

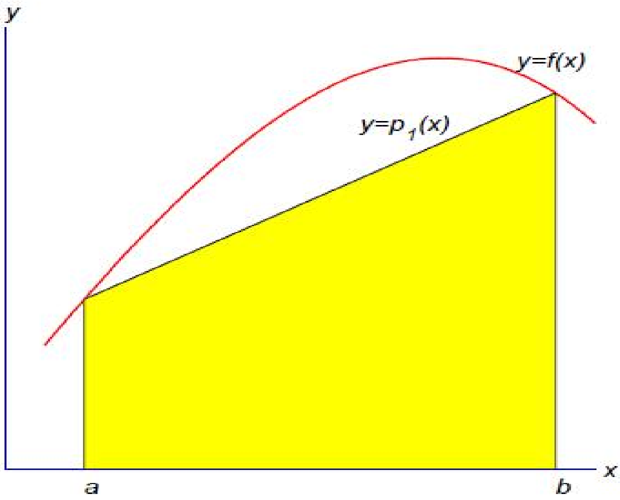

Figure: Illustration of the Trapezoid with one interval to approximate an integral.

Example

Find the approximation $T_1(f)$[3] for the integral. Compute the error and relative error.

$$\int_0^1 \frac{dx}{x^2+1}$$

$$T_1(f) = \frac{b-a}{2}\left[f(a)+f(b)\right]$$

$$a=0, \quad b=1, \quad f(x)=\frac{1}{x^2+1}$$

$$f(0)=\frac{1}{0+1}=1, \quad f(1)=\frac{1}{1+1}=\frac{1}{2}$$

$$T_1(f)=\frac{1-0}{2}\left[1+\frac{1}{2}\right]=\frac{1}{2}\left[3/2\right]=\frac{3}{4}$$

[3]The exact value is $I(f)=\frac{\pi}{4}$.

$$\mathrm{Err}(T_1(f)) = \int_0^1 \frac{dx}{x^2+1} - T_1(f)$$

$$= \frac{\pi}{4} - \frac{3}{4} \doteq 0.0354$$

$$\mathrm{Rel}(T_1(f)) = \frac{\mathrm{Err}(T_1(f))}{\int_0^1 \frac{dx}{x^2+1}} = \frac{\mathrm{Err}(T_1(f))}{\frac{\pi}{4}}$$

$$\doteq 0.0451$$