Section 4.2: Error in Polynomial Interpolation

**Theorem:** For \( n \geq 0 \), suppose \( f \) has \( n + 1 \) continuous derivatives on \([a, b]\) and let \( x_0, \ldots, x_n \) be distinct nodes in \([a, b]\). Then

\[
f(x) - P_n(x) = (x - x_0)(x - x_1) \cdots (x - x_n) \frac{f^{(n+1)}(c_x)}{(n+1)!}
\]

where \( c_x \) is some number between the smallest and largest values of \( x_0, \ldots, x_n \) and \( x \).
Remark about the error formula

The error can be restated as

$$\text{Err}(P_n(x)) = \psi_n(x) \frac{f^{(n+1)}(c_x)}{(n+1)!}$$

where $\psi_n$ is the $n+1$ degree monic polynomial

$$\psi_n(x) = (x - x_0) \cdots (x - x_n) = x^{n+1} + \text{terms with smaller powers}$$

The error depends on the $y$’s due to $f^{(n+1)}(c_x)$, and on the $x$’s due to $\psi_n(x)$. 
Example

Again take \( f(x) = \sin x \) on the interval \([0, \frac{\pi}{2}]\). Let \( 0 \leq x_0 < x_1 < x_2 \leq \frac{\pi}{2} \) and consider the quadratic interpolation \( P_2(x) \). For \( x_0 < x < x_2 \), show that

\[
|f(x) - P_2(x)| \leq \frac{h^3}{9\sqrt{3}}
\]

where \( h = x_1 - x_0 = x_2 - x_1 \).

For \( n = 2 \)

\[
f(x) - P_2(x) = (x-x_0)(x-x_1)(x-x_2) \frac{f'''(c_x)}{3!}
\]

\[
f'''(x) = -\cos x \quad \Rightarrow \quad |f'''(c_x)| = |-\cos(c_x)| \leq 1
\]

\[
\psi_2(x) = (x-x_0)(x-x_1)(x-x_2)
\]
Let \( t = x - x_1 \)

\[
y(t) = (t + h) t (t - h) \\
y = t^3 - h^2 t
\]

To find the maximum value of \( |y| \) find the critical point(s).

\[
y' = 3t^2 - h^2 \quad \Rightarrow \quad y' = 0 \text{ if } 3t^2 - h^2 = 0
\]

\[
y' = 3t^2 - h^2 \quad \Rightarrow \quad t = \frac{\pm h}{\sqrt{3}}
\]

\[
\Rightarrow \quad t^2 = \frac{h^2}{3} \quad \Rightarrow \quad t = \frac{\pm h}{\sqrt{3}}
\]

\[
y \left( \frac{h}{\sqrt{3}} \right) = \left( \frac{h}{\sqrt{3}} \right)^3 - h^2 \left( \frac{h}{\sqrt{3}} \right) = \frac{h^3}{3\sqrt{3}} - \frac{h^3}{\sqrt{3}} = \frac{h^3}{\sqrt{3}} \left( \frac{1}{3} - 1 \right)
\]
\[ y \left( \frac{h}{\sqrt{3}} \right) = -\frac{2h^3}{3\sqrt{3}} \quad \text{and} \quad y \left( \frac{-h}{\sqrt{3}} \right) = \frac{2h^3}{3\sqrt{3}} \]

We have \( |f'''(c_x)| \leq 1 \), \( |\psi_2(x)| \leq \frac{2h^3}{3\sqrt{3}} \)

\[ |f(x) - P_2(x)| = \left| \psi_2(x) \frac{f'''(c_x)}{3!} \right| \leq \frac{2h^3}{3\sqrt{3}} \cdot \frac{1}{6} = \frac{h^3}{9\sqrt{3}} \]
Figure: The maximum value of $|(t + h)t(t - h)|$ occurs at $\pm \frac{h}{\sqrt{3}}$. 
Example

Take \( f(x) = \ln(x + 4) \) on the interval \([-1, 1]\). Let \( x_0 = -1, \ x_1 = 0, \ x_2 = 1 \) and consider the quadratic interpolation \( P_2(x) \). For \(-1 < x < 1\), show that

\[
|f(x) - P_2(x)| \leq \left( \frac{2}{27} \right) \left( \frac{1}{9\sqrt{3}} \right)
\]

\[
\int (x-x_0)(x-x_1)(x-x_2) \frac{f'''(c_x)}{3!}
\]

\[
h = x_1 - x_0 = x_2 - x_1 = 0 - (-1) = 1 - 0 = 1
\]

\[
\therefore \quad |\psi_2(x)| \leq \frac{2h^3}{3\sqrt{3}} = \frac{2}{3\sqrt{3}} \quad \text{for} \quad h = 1
\]
\( f(x) = \ln(x+4) \), \( f'(x) = \frac{1}{x+4} = (x+4)^{-1} \)

\( f''(x) = -(x+4)^{-2} \), \( f'''(x) = 2(x+4)^{-3} = \frac{2}{(x+4)^3} \)

\( f''''(x) = \frac{-6}{(x+4)^4} < 0 \) for \(-1 \leq x \leq 1\)

Since \( f''''(x) \) is positive and decreasing,

\[ |f''''(x)| \leq |f''''(-1)| = \frac{2}{3^3} = \frac{2}{27} \]

So, \( \left| f(x) - P_2(x) \right| \leq \left( \frac{2}{3\sqrt{3}} \right) \frac{2}{27} = \frac{1}{9\sqrt{3}} \left( \frac{2}{27} \right) \)
Behavior of $\Psi(x)$

If we can choose our nodes, an obvious choice is to make them equally spaced. But the question arises:

**Question:** Are equally spaced nodes the best choice for minimizing error?

(Here we’re going to discuss section 4.2.2, then move to section 4.5, and later come back to section 4.3.)
Motivating Example

Suppose we wish to use $P_4(x)$ to interpolate a function $f(x)$ on the interval $[-1, 1]$. We know that the error

$$|f(x) - P_4(x)| = \left| (x - x_0) \cdots (x - x_4) \frac{f^{(5)}(c_x)}{5!} \right| \leq ML$$

where

$$L = \max \left| \frac{f^{(5)}(c_x)}{5!} \right| \quad \text{and} \quad M = \max \left| (x - x_0) \cdots (x - x_4) \right|$$
Motivating Example

Let’s see what kind of control we may have over $M$. We can consider two examples of the function $(x - x_0) \cdots (x - x_4)$.

**Equally Spaced Points:** $\psi_{4,1}(x) = (x + 1)(x + \frac{1}{2})x(x - \frac{1}{2})(x - 1)$

**Not Equally Spaced:**

$\psi_{4,2}(x) = (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)$ where

\[ x_0 = \cos \left( \frac{\pi}{10} \right) \approx 0.9511, \quad x_1 = \cos \left( \frac{3\pi}{10} \right) \approx 0.5878, \]

\[ x_2 = \cos \left( \frac{5\pi}{10} \right) = 0, \quad x_3 = \cos \left( \frac{7\pi}{10} \right) \approx -0.5878, \]

and \[ x_4 = \cos \left( \frac{9\pi}{10} \right) \approx -0.9511 \]
Figure: Two choices of nodes for $P_4$ on $[-1, 1]$. Red dots are equally spaced nodes, and blue dots are an alternative choice (Chebyshev nodes).
Figure: Plot of $\Psi_{4,1}(x)$ and $\Psi_{4,2}(x)$ shows that $\Psi_{4,2}$ has a smaller maximum value.
Motivating Example Continued...

The maximum value of $\Psi_{4,1}(x)$ is $\approx 0.1135$. The maximum value of $\Psi_{4,2}(x)$ is 0.0625.

The error when using equally spaced nodes is 1.8 times as great as the error when using the alternative choice of nodes!
Error for Equally Spaced Nodes

When equally spaced nodes are used, the behavior at the ends (near $a$ and $b$) can be quite dramatic. The error for $x$ in the middle may be small, while the error for $x$ near the ends is much larger.

If $f^{(n+1)}(x)$ is ill behaved, it’s possible that taking $n$ larger results in more error rather than less!

A special case of this is the function

$$f(x) = \frac{1}{1 + x^2} \quad \text{for} \quad -5 \leq x \leq 5$$

(See the next two slides.)
Figure: Plots of $(x - x_0) \cdots (x - x_n)$ for equally spaced nodes on $[0, 1]$ for $n = 2, 4, 6$ and $8$. 
Figure: Plot of $y = \frac{1}{1 + x^2}$ (red) together with degree 10 polynomial interpolation $P_{10}(x)$ (blue dash) obtained using equally spaced nodes on $[-5, 5]$. 
Definition: For an integer $n \geq 0$ define the function

$$T_n(x) = \cos \left( n \cos^{-1}(x) \right), \quad -1 \leq x \leq 1.$$ 

It can be shown that $T_n$ is a polynomial of degree $n$. It’s called the **Chebyshev Polynomial of degree $n$**.
Chebyshev Polynomials

Determine the polynomials $T_0(x)$, $T_1(x)$, and $T_2(x)$ in the form of ordinary polynomials.

$$T_n(x) = \cos \left( n \cos^{-1} x \right), \quad -1 \leq x \leq 1$$

$n = 0,$

$$T_0(x) = \cos(0) = 1$$

$n = 1,$

$$T_1(x) = \cos \left( \cos^{-1} x \right) = x$$

$n = 2,$

$$T_2(x) = \cos \left( 2 \cos^{-1} x \right)
= 2 \cos^2 \left( \cos^{-1} x \right) - 1$$

$$T_2(x) = 2x^2 - 1$$
* \( \cos(2\theta) = 2\cos^2\theta - 1 \) *

\[ T_0(x) = 1 \quad \text{if} \quad -1 \leq x \leq 1 \]

\[ T_1(x) = x \]

\[ T_2(x) = 2x^2 - 1 \]
Recursion Relation

\[ T_0(x) = 1 \text{ and } T_1(x) = x. \] It can be shown that for \( n \geq 1 \)

\[ T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x). \]

Compute \( T_2(x) \) and \( T_3(x) \) using this relation.

\[ T_2(x) = 2x T_1(x) - T_0(x) = 2x(x) - 1 = 2x^2 - 1 \]

\[ T_3(x) = 2x T_2(x) - T_1(x) = 2x \left(2x^2 - 1\right) - x \]

\[ = 4x^3 - 2x - x = 4x^3 - 3x \]

\[ T_2(x) = 2x^2 - 1 \quad T_3(x) = 4x^3 - 3x \]
Figure: Plot of the first six Chebyshev Polynomials (of the first kind).
Some Properties of Chebyshev Polynomials

- $T_n$ is an even function if $n$ is even and an odd function if $n$ is odd.
- $T_n(1) = 1$ and $T_n(-1) = (-1)^n$ for every $n$.
- They have an orthogonality relation

$$
\int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1 - x^2}} \, dx = 0 \quad n \neq m.
$$

- And the main property we’re interested in

$$
|T_n(x)| \leq 1 \quad \text{for all } -1 \leq x \leq 1.
$$
Minimum Size Property

We can note that

\[ T_n(x) = 2^{n-1} x^n + \text{terms with lower powers}. \]

We define the modified Chebyshev polynomials by

\[ \tilde{T}_n(x) = \frac{1}{2^{n-1}} T_n(x). \]

Remark: The modified Chebyshev polynomials are monic polynomials. That is

\[ \tilde{T}_n(x) = x^n + \text{terms with lower powers}. \]
Minimum Size Property

**Theorem:** Let $n \geq 1$ be an integer. Of all monic polynomials on the interval $[-1, 1]$, the one with the smallest maximum value is the modified Chebyshev polynomial $\tilde{T}_n(x)$. Moreover

$$|\tilde{T}_n(x)| \leq \frac{1}{2^{n-1}} \text{ for all } -1 \leq x \leq 1.$$ 

This result suggests that whenever possible, we choose the polynomial $\psi_n(x)$ in our error theorem to be the modified Chebyshev polynomial $\tilde{T}_{n+1}(x)$. 


Chebyshev Nodes

Since \( \tilde{T}_{n+1}(x) \) is monic, it can be written as

\[
\tilde{T}_{n+1}(x) = (x - r_0)(x - r_1) \cdots (x - r_n)
\]

where \( r_0, \ldots, r_n \) are the roots of \( T_{n+1}(x) \).

We had the polynomial in our error formula

\[
\psi_n(x) = (x - x_0)(x - x_1) \cdots (x - x_n).
\]

So to minimize the error—i.e. make \( \psi_n(x) = \tilde{T}_{n+1}(x) \)—we would have to

choose the nodes \( x_j \) to be the roots \( r_j \) of the Chebyshev polynomial \( T_{n+1} \).
Example: Chebyshev Nodes

Use the change of variables $x = \cos \theta$ to find the roots of $T_4(x)$.

$$T_4(x) = \cos \left( 4 \cos^{-1}(x) \right)$$

Recall: \( \cos \phi = 0 \) if \( \phi = \frac{\pi}{2} + j\pi \) for \( j = 0, \pm 1, \pm 2, \ldots \)

\[
\phi = \frac{\pi}{2} + \frac{2j\pi}{2} = \left(1 + 2j\right)\pi
\]

\[
T_4(x) = \cos \left( 4 \cos^{-1}x \right) = \cos \left( 4\Theta \right) \quad \text{for} \quad \Theta = \cos^{-1}x
\]

so \( T_4(x) = 0 \) if

\[
4\Theta = \frac{(2j+1)\pi}{2} \quad \text{for} \quad j = 0, \pm 1, \pm 2, \ldots
\]
So \[ \theta = \frac{(2j+1)\pi}{2^j} = \frac{(2j+1)\pi}{8}, \quad j = 0, 1, 2, 3\]

(There are 4 distinct zeros.)

\[ \theta_0 = \frac{\pi}{8}, \quad \theta_1 = \frac{3\pi}{8}, \quad \theta_2 = \frac{5\pi}{8}, \quad \theta_3 = \frac{7\pi}{8} \]

Since \[ x = \cos \theta \]

\[ x_0 = \cos \frac{\pi}{8}, \quad x_1 = \cos \frac{3\pi}{8}, \quad x_2 = \cos \frac{5\pi}{8}, \quad x_3 = \cos \frac{7\pi}{8} \]