Mar. 19 Math 2254H sec 015H Spring 2015

Section 11.2: Series

Observation: The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, but the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent.

Both series share the property that the n^{th} term goes to zero. That is, both

$$\lim_{n\to\infty} \frac{1}{n} = 0, \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

Theorem: (a test for divergence)

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Caution: The converse is NOT true!

Theorem: (The Divergence Test)[1] If

$$\lim_{n\to\infty} a_n \quad \text{does not exists, or} \qquad \lim_{n\to\infty} a_n \neq 0,$$

then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

[1] The Divergence Test is also known as the $\mathbf{n}^{th}$ **Term Test**.

Example:

If possible, determine if the series is convergent or divergent. If it is not possible to determine if the series converges, explain why.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n+3}$ Divergence test $a_n = \frac{2n}{n+3}$

$$\lim_{n\to\infty} \frac{2n}{n+3} = \lim_{n\to\infty} \frac{2n}{n+3} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{2}{1+\frac{3}{n}} = \frac{2}{1+0} = 2 \neq 0$$

The series diverges by the divergence test.

Examples continued...

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Divergence test : $a_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

The test fails.

No conclusion can be reached

w/ present "tests."

Examples continued...

(c) $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n}$

Divergence test $\quad a_n = \left(1+\frac{1}{n}\right)^{n}$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e \neq 0$$

The series is divergent.

Theorem: Some Properties of Convergent Series

Theorem: Suppose $\sum a_n$ and $\sum b_n$ are convergent series with sums α and β, respectively. Then the series

$$\sum (a_k + b_k), \quad \sum (a_k - b_k), \quad \text{and} \quad \sum c a_k \ \text{ for constant } c$$

are convergent with sums

$$\sum (a_k + b_k) = \alpha + \beta, \quad \sum (a_k - b_k) = \alpha - \beta,$$

$$\text{and} \quad \sum c a_k = c\alpha.$$

Example

Find the sum of the series

$$\sum_{n=1}^{\infty}\left(\frac{4}{n(n+1)}+\frac{2}{5^{n-1}}\right) = 4 + \frac{5}{2} = \frac{8+5}{2} = \frac{13}{2}$$

$$\sum_{n=1}^{\infty} \frac{4}{n(n+1)} = 4\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 4\cdot 1 = 4$$

$$\sum_{n=1}^{\infty} \frac{2}{5^{n-1}} = \sum_{n=1}^{\infty} 2\left(\frac{1}{5}\right)^{n-1} = \frac{2}{1-\frac{1}{5}}\cdot\frac{5}{5} = \frac{10}{4} = \frac{5}{2}$$

geometric

$a=2$

$r=\frac{1}{5}$

Section 11.3: The Integral Test

Recall: Integrals were defined in terms of sums—Riemann Sums—and there is a geometric way, relating to area between curves, to interpret them.

Note: A series can be related to areas too

$$a_1 + a_2 + \cdots = a_1 \cdot 1 + a_2 \cdot 1 + \cdots$$

if the numbers a_k are heights and all the widths are 1. Of course, this makes best sense when the numbers a_k are positive.

Context for this Section: We will restrict our attention <u>for the moment</u> to series of nonnegative terms.

Relating an Integral to a Series (divergent)

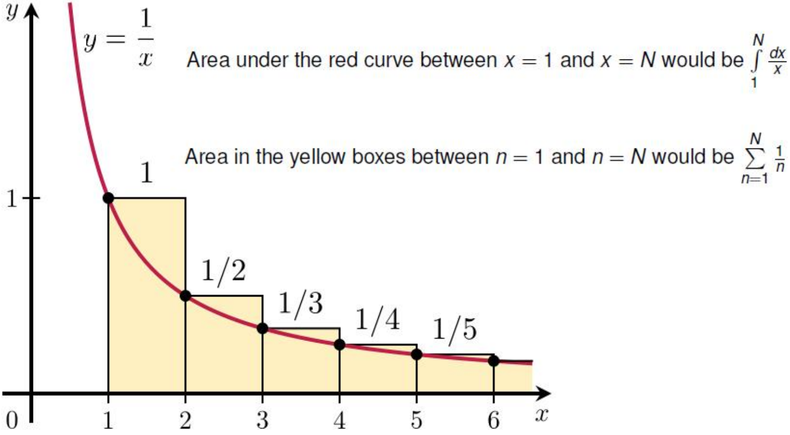


Figure: Comparison of *areas* related to $\int_1^\infty \frac{dx}{x}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$.

Relating an Integral to a Series (convergent)

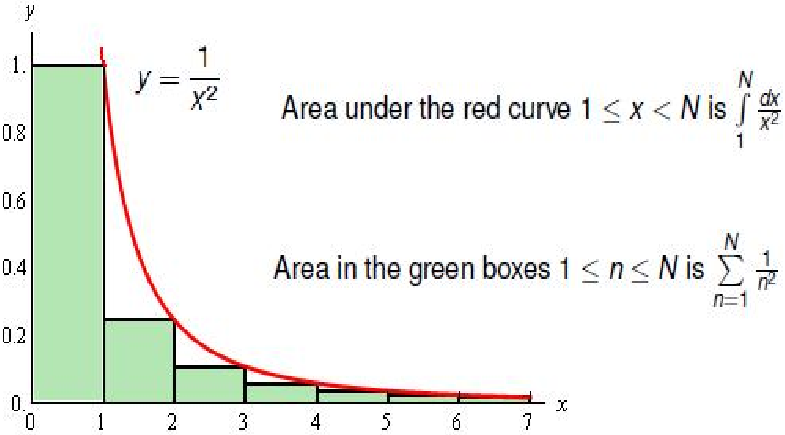


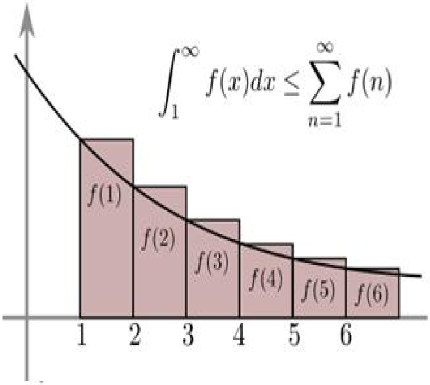
Figure: Comparison of *areas* related to $\int_1^\infty \frac{dx}{x^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Set Up for the Integral Test

Question: Does the series of positive terms $\sum_{n=1}^{\infty} a_n$ converge or diverge?

- Suppose f is a continuous, positive, decreasing function defined on the interval $[1, \infty)$.
- Also suppose that $a_n = f(n)$—the function and the terms in the series have the same "formula".
- Assume that we are able to determine if the integral $\int_1^{\infty} f(x)\,dx$ converges or diverges.

Geometric Interpretation of the Integral Test



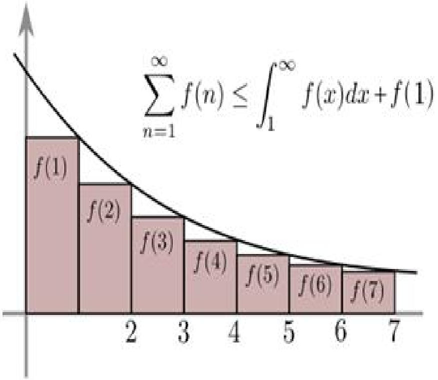


Figure: The possible value of the series can be trapped between the possible values of integrals.

The Integral Test

Theorem: Let $\sum a_n$ be a series of positive terms and let the function f defined on $[1, \infty)$ be continuous, positive and decreasing with

$$a_n = f(n).$$

(i) If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^\infty a_n$ is convergent.

(ii) If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^\infty a_n$ is divergent.

Both series and integral converge, or both series and integral diverge.

Examples:

Determine the convergence or divergence of the series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Integral test: Let $f(x) = \dfrac{1}{x^2+1}$

f is positive, continuous and decreasing.

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{dx}{x^2+1} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^2+1}$$

$$= \lim_{t\to\infty} \tan^{-1}x \Big|_1^t$$

$$= \lim_{t\to\infty} \left(\tan^{-1} t - \tan^{-1} 1 \right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

The integral converges.

The series must also converge.

Examples:

Integral test : $f(x) = \frac{\ln x}{x}$, $x \geq 1$

(b) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

f is continuous , positive for $x > 1$,

$$f'(x) = \frac{\frac{1}{x} \cdot x - 1 \cdot \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

$f'(x) < 0$ if $1 - \ln x < 0 \Rightarrow 1 < \ln x$

for $x > e$

we'll consider $\sum_{n=3}^{\infty} \frac{\ln n}{n}$

for $x \geq 3$, f is continuous, positive, decreasing.

$u = \ln x$

$\int u\,du = \frac{u^2}{2} + C$

$$\int_3^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_3^t \frac{\ln x}{x}\,dx$$

$$= \lim_{t\to\infty} \frac{1}{2}(\ln x)^2 \Big|_3^t$$

$$= \lim_{t\to\infty} \left[\frac{1}{2}(\ln t)^2 - \frac{1}{2}(\ln 3)^2\right] = \infty$$

The integral diverges, so $\sum_{n=3}^\infty \frac{\ln n}{n}$ diverges.

Hence the original series $\sum_{n=1}^\infty \frac{\ln n}{n}$ diverges.

Special Series: *p*-series

Determine the values of p for which the series converges.

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

$\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p>1$

and

diverges if $p \le 1$.

For the integral test, $f(x) = \frac{1}{x^p}$

is positive, continuous, decreasing

for $p > 0$.

So by the integral test,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges if} \quad p>1$$

and diverges if $\quad p \leq 1$.