March 1 Math 2306 sec. 57 Spring 2018

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

Constant coef. left   Polynomial right

Here $g(x) = 8x+1$, a 1^{st} degree polynomial.
We'll "guess" that y_p is also a 1^{st} degree polynomial.

Suppose $y_p = Ax+B$ for some <u>constants</u> A, B.

This is supposed to solve the ODE. So substitute

$y_p' = A$, $y_p'' = 0$

Need $\quad y_p'' - 4y_p' + 4y_p = 8x+1$

$$0 - 4(A) + 4(Ax+B) = 8x+1$$

Collect $\quad 4Ax + (-4A+4B) = 8x+1$

Match coefficients

$$\left.\begin{aligned} 4A &= 8 \\ -4A+4B &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} &A = 2 \\ &B = \frac{1}{4}(1+4A) = \frac{1}{4}(1+8) = \frac{9}{4} \end{aligned}$$

So the particular solution

$$\boxed{y_p = 2x + \frac{9}{4}}$$

The Method: Assume y_p has the same **form** as $g(x)$

$$y'' - 4y' + 4y = 6e^{-3x}$$

Constant coefficient left. Exponential right.

$g(x) = 6e^{-3x}$ a constant times e^{-3x}

Guess $y_p = Ae^{-3x}$ for some constant A.

Substitute $y_p' = -3Ae^{-3x}$, $y_p'' = 9Ae^{-3x}$

$y_p'' - 4y_p' + 4y_p = 6e^{-3x}$

$9Ae^{-3x} - 4(-3Ae^{-3x}) + 4Ae^{-3x} = 6e^{-3x}$

$25Ae^{-3x} = 6e^{-3x}$

$A = \frac{6}{25}$

$\Rightarrow \boxed{y_p = \frac{6}{25}e^{-3x}}$

Make the form general

$$y'' - 4y' + 4y = 16x^2$$

Suppose we look @ $g(x) = 16x^2$ and see "a constant times x^2." Let's try $y_p = Ax^2$. Substitute

$y_p' = 2Ax$, $y_p'' = 2A$.

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax) + 4(Ax^2) = 16x^2$$

$$4Ax^2 - 8Ax + 2A = 16x^2$$

Matching coefficients

$$\left.\begin{array}{l} 4A = 16 \\ -8A = 0 \end{array}\right\} \Rightarrow \quad A = 4 \ \underline{\text{and}} \ A = 0$$

not possible

y_p can't look like Ax^2.

We need to consider $g(x) = 16x^2$ as a 2^{nd} degree polynomial and guess

$$y_p = Ax^2 + Bx + C \qquad A, B, C \text{ constants}$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax+B) + 4(Ax^2+Bx+C) = 16x^2 + 0x + 0$$

$$\underline{4A}x^2 + (\underline{-8A+4B})x + (\underline{2A-4B+4C}) = \underline{16}x^2 + \underline{0}x + \underline{0}$$

$$4A = 16 \Rightarrow A = 4$$

$$-8A + 4B = 0 \Rightarrow B = \frac{8}{4}A = 2A = 8$$

$$2A - 4B + 4C = 0 \Rightarrow C = \frac{4}{4}B - \frac{2}{4}A = 8 - \frac{1}{2}(4) = 6$$

So

$$\boxed{y_p = 4x^2 + 8x + 6}$$

General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

If we assume that $y_p = A\sin(2x)$, taking two derivatives would lead to the equation

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

This would require (matching coefficients of sines and cosines)

$$-4A = 20 \quad \text{and} \quad -2A = 0.$$

This is impossible as it would require $-5 = 0$!

General Form: sines and cosines

We must think of our equation $y'' - y' = 20\sin(2x)$ as

$$y'' - y' = 20\sin(2x) + 0\cos(2x).$$

The correct format for y_p is

$$y_p = A\sin(2x) + B\cos(2x).$$

Examples of Forms of y_p based on g (Trial Guesses)

(a) $g(x) = 1$ (or really any nonzero constant)

$y_p = A$ also a constant

(b) $g(x) = x - 7$ 1st degree poly.

$y_p = Ax + B$

Examples of Forms of y_p based on g (Trial Guesses)

(c) $g(x) = 5x^2$ 2nd degree poly

$$y_p = Ax^2 + Bx + C$$

(d) $g(x) = 3x^3 - 5$ 3rd degree poly

$$y_p = Ax^3 + Bx^2 + Cx + D$$

Examples of Forms of y_p based on g (Trial Guesses)

(e) $g(x) = xe^{3x}$

1st degree poly times e^{3x}

$$y_p = (Ax+B)e^{3x} = Axe^{3x} + Be^{3x}$$

(f) $g(x) = \cos(7x)$

Linear combination of $\cos(7x)$ and $\sin(7x)$

$$y_p = A\cos(7x) + B\sin(7x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(g) $g(x) = \sin(2x) - \cos(4x)$

Linear combo of sin(2x) and cos(2x) and sin(4x) and cos(4x)

$$y_p = A\sin(2x) + B\cos(2x) + C\cos(4x) + D\sin(4x).$$

(h) $g(x) = x^2\sin(3x)$

2^{nd} degree poly times sin(3x) and 2^{nd} degree poly times cos(3x)

$$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x).$$

Examples of Forms of y_p based on g (Trial Guesses)

(i) $g(x) = e^x \cos(2x)$ $\quad$ $e^x \cos(2x)$ and $e^x \sin(2x)$

$$y_p = Ae^x \cos(2x) + Be^x \sin(2x)$$

(j) $g(x) = xe^{-x} \sin(\pi x)$ $\quad$ 1st degree poly times e^{-x} times sine and cosine of πx.

$$y_p = (Ax+B)e^{-x} \sin(\pi x) + (Cx+D)e^{-x} \cos(\pi x)$$

The Superposition Principle

Example: Determine the correct form of the particular solution using the method of undetermined coefficients for the ODE

$$y'' - 4y' + 4y = 6e^{-3x} + 16x^2$$

Superposition tells us we can write $y_p = y_{p_1} + y_{p_2}$

where y_{p_1} solves

$$y'' - 4y' + 4y = 6e^{-3x}$$

$y_{p_1} = Ae^{-3x}$

and y_{p_2} solves

$$y'' - 4y' + 4y = 16x^2$$

$y_{p_2} = Bx^2 + Cx + D$

So $y_p = Ae^{-3x} + Bx^2 + Cx + D$

we found before that $y_{p_1} = \frac{6}{25} e^{-3x}$ and

$$y_{p_2} = 4x^2 + 8x + 6$$

So $y_p = \frac{6}{25} e^{-3x} + 4x^2 + 8x + 6$

A Glitch!

$$y'' - y' = 3e^x$$

Constant coef. exponential

$g(x) = 3e^x$, guess $y_p = Ae^x$

Substitute $y_p' = Ae^x$ and $y_p'' = Ae^x$

$y_p'' - y_p' = Ae^x - Ae^x = 3e^x$

$0 = 3e^x$

$0 = 3$??

y_p can't have the form Ae^x.

Consider the associated homogeneous equation

$$y'' - y' = 0$$

Characteristic eqn $\quad m^2 - m = 0$

$m(m-1) = 0 \implies m = 0$ or $m = 1$

$$y_1 = e^{0x} = 1 \quad , \quad y_2 = e^{1x} = e^{x}$$

$g(x)$ and our guess @ y_p are part of y_c. We'll need a way to modify our guess @ y_p.

We'll consider cases

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply that term by x^n, where n is the smallest positive integer that eliminates the duplication.