March 1 Math 2306 sec 58 Spring 2016

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

We seek solutions of the form $y = e^{mx}$ for constant m, and obtain the characteristic (a.k.a. auxiliary) equation

$$am^2 + bm + c = 0.$$

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_{1,2} = \alpha \pm i\beta$

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \text{where} \quad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note that $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ form a fundamental solution set.

Example

Find the general solution of the ODE

$$y'' - 2y' - 2y = 0$$

Characteristic equation :

$$m^2 - 2m - 2 = 0$$

$$m = \frac{2 \pm \sqrt{2^2 - 1\cdot 4(-2)}}{2} = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2}$$

$$= 1 \pm \sqrt{3}$$

$m_1 = 1+\sqrt{3}, \quad m_2 = 1-\sqrt{3} \quad$ so

$$y_1 = e^{(1+\sqrt{3})x}, \quad y_2 = e^{(1-\sqrt{3})x}$$

The general solution is

$$y = c_1 e^{(1+\sqrt{3})x} + c_2 e^{(1-\sqrt{3})x}$$

Example

Solve the IVP

$$y'' + y' - 12y = 0, \quad y(0) = 1, \quad y'(0) = 10$$

Characteristic eqn: $m^2 + m - 12 = 0$

factor $(m+4)(m-3) = 0$

$m_1 = -4, \quad m_2 = 3$

$y_1 = e^{-4x}, \quad y_2 = e^{3x}$

The general solution is $y = c_1 e^{-4x} + c_2 e^{3x}$

Apply $y(0)=1$, $y'(0)=10$

$$y = c_1 e^{-4x} + c_2 e^{3x}$$

$$y' = -4c_1 e^{-4x} + 3c_2 e^{3x}$$

$$y(0) = c_1 e^0 + c_2 e^0 = 1$$

$$y'(0) = -4c_1 e^0 + 3c_2 e^0 = 10$$

$$\left.\begin{array}{l} c_1 + c_2 = 1 \\ -4c_1 + 3c_2 = 10 \end{array}\right\} \Rightarrow \begin{array}{r} 4c_1 + 4c_2 = 4 \\ -4c_1 + 3c_2 = 10 \\ \hline 7c_2 = 14 \end{array}$$

add

$$7c_2 = 14 \Rightarrow c_2 = 2$$

$$c_1 + c_2 = 1 \Rightarrow c_1 = 1 - c_2 = 1 - 2 = -1$$

The solution to the IVP is

$$y = -e^{-4x} + 2e^{3x}.$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

$$y = c_1 e^{mx} + c_2 x e^{mx} \quad \text{where} \quad m = \frac{-b}{2a}$$

Use reduction of order to show that if $y_1 = e^{\frac{-bx}{2a}}$, then $y_2 = xe^{\frac{-bx}{2a}}$.

Standard form $\quad y'' + \frac{b}{a} y' + \frac{c}{a} y = 0$

$P(x) = \frac{b}{a} \qquad y_2 = u(x)\, y_1(x) \quad$ where

$$u = \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2}\, dx$$

$$u = \int \frac{e^{-\int \frac{b}{a}\,dx}}{\left(e^{-\frac{b}{2a}x}\right)^2}\,dx = \int \frac{e^{-\frac{b}{a}x}}{e^{-\frac{2b}{2a}x}}\,dx$$

$$= \int \frac{e^{-\frac{b}{a}x}}{e^{-\frac{b}{a}x}}\,dx = \int dx = x$$

So $\quad y_2 = x\,e^{-\frac{b}{2a}x}$

Example

Solve the ODE

$$4y'' - 4y' + y = 0$$

Characteristic eqn $\quad 4m^2 - 4m + 1 = 0$

$$(2m-1)^2 = 0$$

$m = \frac{1}{2}$ repeated root.

So $\quad y_1 = e^{\frac{1}{2}x}, \quad y_2 = x e^{\frac{1}{2}x}$

The general solution is

$$y = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}.$$

Example

Solve the IVP

$$y'' + 6y' + 9y = 0, \quad y(0) = 4, \quad y'(0) = 0$$

Characteristic eqn. $m^2 + 6m + 9 = 0$

$$(m+3)^2 = 0$$

$m = -3$, repeated root

$y_1 = e^{-3x}$, $y_2 = xe^{-3x}$

General solution $y = c_1 e^{-3x} + c_2 x e^{-3x}$

Apply $y(0)=4$, $y'(0)=0$

$$y(0)=C_1e^0+C_2\cdot 0e^0=4$$

$$C_1=4$$

$$y=C_1e^{-3x}+C_2xe^{-3x}$$

$$y'=-3C_1e^{-3x}+C_2e^{-3x}-3C_2xe^{-3x}$$

$$y'(0)=-3C_1e^0+C_2e^0-3C_2\cdot 0e^0=0$$

$$-12+C_2=0 \implies C_2=12$$

The solution to the IVP is

$$y=4e^{-3x}+12xe^{-3x}.$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)), \quad \text{where the roots}$$

$$m = \alpha \pm i\beta, \quad \alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

The solutions can be written as

$$Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}.$$

Deriving the solutions Case III

Recall Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$Y_1 = e^{\alpha x} e^{i\beta x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right)$$

$$Y_2 = e^{\alpha x} e^{-i\beta x} = e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right)$$

$$Y_1 = e^{\alpha x}\cos(\beta x) + i\, e^{\alpha x}\sin(\beta x)$$

$$Y_2 = e^{\alpha x}\cos(\beta x) - i\, e^{\alpha x}\sin(\beta x)$$

Using the principle of superposition

Set $y_1 = \frac{1}{2}Y_1 + \frac{1}{2}Y_2 = \frac{1}{2}(Y_1 + Y_2)$

$$= \frac{1}{2}\left(2e^{\alpha x}\cos\beta x\right) = e^{\alpha x}\cos\beta x$$

$$y_2 = \frac{1}{2i}Y_1 - \frac{1}{2i}Y_2 = \frac{1}{2i}(Y_1 - Y_2)$$

$$= \frac{1}{2i}\left(2i\,e^{\alpha x}\sin\beta x\right) = e^{\alpha x}\sin\beta x$$

Our fundamental solution set

is

$$y_1 = e^{\alpha x} \cos \beta x$$

$$y_2 = e^{\alpha x} \sin \beta x$$

Example

Solve the ODE

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 6x = 0$$

Characteristic eqn. $\quad m^2 + 4m + 6 = 0$

$$m = \frac{-4 \pm \sqrt{4^2 - 1\cdot 4\cdot 6}}{2} = \frac{-4 \pm \sqrt{16-24}}{2} = \frac{-4 \pm \sqrt{-8}}{2}$$

$$= \frac{-4 \pm i2\sqrt{2}}{2} = -2 \pm i\sqrt{2} \qquad \alpha \pm i\beta$$

$\alpha = -2$ and $\beta = \sqrt{2}$

$$x_1 = e^{-2t}\cos(\sqrt{2}\,t) \quad \text{and} \quad x_2 = e^{-2t}\sin(\sqrt{2}\,t)$$

The general solution is

$$x = c_1 e^{-2t}\cos(\sqrt{2}\,t) + c_2 e^{-2t}\sin(\sqrt{2}\,t)$$

Example

Solve the IVP

$$y'' + 4y = 0, \quad y(0) = 3, \quad y'(0) = -5$$

Characteristic eqn. $\quad m^2 + 4 = 0$

$m^2 = -4 \quad \Rightarrow \quad m = \pm\sqrt{-4} = \pm 2i$

$\alpha \pm i\beta$

$\alpha = 0$ and $\beta = 2$

$y_1 = e^{0x}\cos(2x) = \cos(2x)$, $y_2 = e^{0x}\sin(2x) = \sin(2x)$

General solution is $y = c_1 \cos(2x) + c_2 \sin(2x)$

Apply $y(0) = 3$, $y'(0) = -5$

$y = c_1 \cos(2x) + c_2 \sin(2x)$, $\quad y(0) = c_1 \cos 0 + c_2 \sin 0 = 3$

$c_1 = 3$

$y' = -2c_1 \sin(2x) + 2c_2 \cos(2x)$

$y'(0) = -2c_1 \sin 0 + 2c_2 \cos 0 = -5$

$2c_2 = -5 \Rightarrow c_2 = \frac{-5}{2}$

$$\boxed{\text{The solution to the IVP is } \quad y = 3\cos(2x) - \frac{5}{2}\sin(2x).}$$

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.
- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$.
- If a root m is repeated k times, we get k linearly independent solutions $(\hat{m}-m)^k$

$$e^{mx}, \quad xe^{mx}, \quad x^2e^{mx}, \quad \ldots, \quad x^{k-1}e^{mx}$$

  or in conjugate pairs cases $2k$ solutions

$$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x), \ldots,$$

$$x^{k-1}e^{\alpha x}\cos(\beta x),\ x^{k-1}e^{\alpha x}\sin(\beta x)$$

- It may require a computer algebra system to find the roots for a high degree polynomial.

Example

Solve the ODE

$y''' - 4y' = 0$

3rd order

Characteristic equation is

$$m^3 - 4m = 0$$

factor $\quad m(m^2-4) = 0$

$$m(m-2)(m+2) = 0$$

Every fundamental solution set must have 3 solutions.

$$m_1 = 0, \quad m_2 = 2, \quad m_3 = -2$$

$$y_1 = e^{0x} = 1 \qquad y_2 = e^{2x}, \quad y_3 = e^{-2x}$$

The general solution is

$$y = c_1 + c_2 e^{2x} + c_3 e^{-2x}$$

Note $y = c_1 y_1 + c_2 y_2 + c_3 y_3$.