March 1 Math 2306 sec 58 Spring 2016

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

$$arac{d^2y}{dx^2}+brac{dy}{dx}+cy=0.$$

We seek solutions of the form $y = e^{mx}$ for constant *m*, and obtain the characteristic (a.k.a. auxiliary) equation

$$am^2+bm+c=0.$$

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Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

- $b^2 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$
- II $b^2 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$
- III $b^2 4ac < 0$ and there are two roots that are complex conjugates $m_{1,2} = \alpha \pm i\beta$

Case I: Two distinct real roots

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac > 0$
 $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ where $m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Note that $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ form a fundamental solution set.

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Find the general solution of the ODE

$$y^{\prime\prime}-2y^{\prime}-2y=0$$

$$m^2 - 2m - 2 = 0$$

$$m = \frac{2 \pm \sqrt{2^2 - 1.4(-2)}}{2} = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2}$$

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Solve the IVP

$$y'' + y' - 12y = 0, \quad y(0) = 1, \quad y'(0) = 10$$

Characteristic eqn: $m^{2} + m - 12 = 0$
factor $(m+4)(m-3) = 0$
 $m_{1} = -4, \quad m_{2} = 3$
 $y_{1} = e^{-4x}, \quad y_{2} = e^{-4x}$
The general solution is $y = c_{1}e^{-4x} + c_{2}e^{-4x}$

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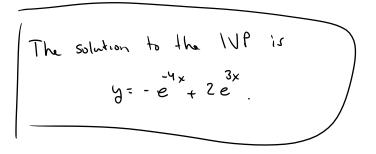
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$$\begin{array}{l} App^{1} & \mathcal{Y}(0)=1, \ \mathcal{Y}^{1}(0)=10 \\ & \mathcal{Y}=C_{1}e^{4\chi}+C_{2}e^{2\chi} \\ & \mathcal{Y}=-4C_{1}e^{4\chi}+3C_{2}e^{3\chi} \\ & \mathcal{Y}^{1}(0)=-4C_{1}e^{2\chi}+3C_{2}e^{2\chi} \\ & \mathcal{Y}^{1}(0)=-4C_{1}e^{2\chi}+3C_{2}e^{2\chi}=10 \\ &$$

 $C_1 + C_2 = | \Rightarrow C_1 = | - C_2 = | - 2 = -|$

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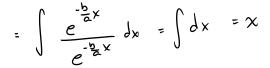
Case II: One repeated real root

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac = 0$
 $y = c_1 e^{mx} + c_2 x e^{mx}$ where $m = \frac{-b}{2a}$

Use reduction of order to show that if $y_1 = e^{\frac{-bx}{2a}}$, then $y_2 = xe^{\frac{-bx}{2a}}$. Standard form $y'' + \frac{b}{a}y' + \frac{c}{a}y = 0$ $P(x) = \frac{b}{a}$ $y_2 = u(x)y_1(x)$ where $u = \int \frac{-\int P(wdx)}{(y_1(x))^2} dx$

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$$u = \int \frac{-\int \frac{b}{a} dx}{\left(e^{\frac{-b}{2a}x}\right)^2} dx = \int \frac{-\frac{b}{a}x}{\frac{e}{2a}x} dx$$



So
$$y_2 = X e$$

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Solve the ODE

$$4y''-4y'+y=0$$

Charadaistic eqn
$$4m^2 - 4m + 1 = 0$$

 $(2m - 1)^2 = 0$
 $m = \frac{1}{2}$ repeated root.

so
$$y_1 = e^{\frac{1}{2}x}$$
, $y_2 = xe^{\frac{1}{2}x}$

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The general solution is

$$y = C_1 e^{\frac{1}{2}x} + C_2 x e^{\frac{1}{2}x}$$

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Solve the IVP

$$y'' + 6y' + 9y = 0, \quad y(0) = 4, \quad y'(0) = 0$$

Charaduistic eqn. $m^2 + 6m + 9 = 0$
 $(m+3)^2 = 0$
 $m = -3, \quad repealed \quad root$

$$\begin{array}{l} \text{Apply } y_{0}(0) = 4, \ y'(0) = 0 \\ y = c_{1}e^{-3x} + c_{2} \times e^{3x} \\ y = c_{1}e^{-3x} + c_{2} \times e^{-3x} \\ y' = -3c_{1}e^{-3x} + c_{2}e^{-3x} - 3c_{2} \times e^{-3x} \\ y'_{0}(0) = -3c_{1}e^{2} + c_{2}e^{-3c_{1}} \cdot 0e^{0} = 0 \\ -12 + (z = 0) \Rightarrow c_{2} = 12 \end{array}$$

$$\begin{array}{l} \text{The solution } h + h \quad |VP | i(z) \\ y = 4e^{-3x} + 12 \times e^{-3x} \\ y = 4e^{-3x} + 12 \times e^{-3x} \\ y = 12$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac < 0$
 $y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$, where the roots $m = \alpha \pm i\beta$, $\alpha = \frac{-b}{2a}$ and $\beta = \frac{\sqrt{4ac - b^2}}{2a}$

The solutions can be written as

$$Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}$$
, and $Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}$.

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Deriving the solutions Case III

Recall Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$Y_{1} = e^{ax} e^{i\beta x} = e^{ax} \left(C_{os}(\beta x) + i Sin(\beta x) \right)$$

$$Y_{2} = e^{ax} e^{i\beta x} = e^{ax} \left(Cos(\beta x) - i Sin(\beta x) \right)$$

$$Y_{1} = e^{ax} C_{os}(\beta x) + i e^{ax} Sin(\beta x)$$

$$Y_{2} = e^{ax} Cos(\beta x) - i e^{ax} Sin(\beta x)$$

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Using the principle of superposition
Set
$$y_1 = \frac{1}{2}Y_1 + \frac{1}{2}Y_2 = \frac{1}{2}(Y_1 + Y_2)$$

 $= \frac{1}{2}(2e^{dx}\cos\beta x) = e^{dx}\cos\beta x$

$$y_{2} = \frac{1}{2i} (Y_{1} - \frac{1}{2i} (Y_{2} = \frac{1}{2i} (Y_{1} - Y_{2}))$$
$$= \frac{1}{2i} (2ie^{ix} Sin \beta x) = e^{ix} Sin \beta x$$

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Our fundamental solution set is y, = e^x Cos px yz = e^x Sinpx



Solve the ODE

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 6x = 0$$

$$M = -\frac{4 \pm \sqrt{4^{2} - 1.4.6}}{2} = -\frac{4 \pm \sqrt{16 - 24}}{2} = -\frac{4 \pm \sqrt{-8}}{2}$$

d = -2 and B=12

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$$X_{1} = e^{-2t} C_{0J} (J\overline{z}t) \quad and \quad X_{2} = e^{-2t} Sin(J\overline{z}t)$$

$$The general Solution is$$

$$X = C_{1} e^{-2t} C_{0J} (J\overline{z}t) + C_{2} e^{-2t} Sin(J\overline{z}t)$$

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Solve the IVP

$$y'' + 4y = 0, \quad y(0) = 3, \quad y'(0) = -5$$

Characteristic eqn. $m^2 + 4 = 0$
 $m^2 = -4 \quad \Rightarrow \quad m = \pm \sqrt{-4} = \pm 2i$
 $q \pm i\beta$
 $q \equiv 0 \quad \text{ond} \quad \beta \equiv 2$
 $y_1 = e^{0x} C_{0s}(2x) = C_{0s}(2x), \quad y_2 = e^{0x} Sin(2x) = Sin(2x)$

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General Solution is
$$y = C_1 Cos(2x) + C_2 Sin(2x)$$

Apply $y(0) = 3$, $y'(0) = -5$
 $y = C_1 Cos(2x) + C_2 Sin(2x)$, $y(0) = C_1 CosO + C_2 SinO = 3$
 $y' = -2C_1 Sin(2x) + 2C_2 Cos(2x)$
 $y'(0) = -2C_1 SinO + 2C_2 CosO = -5$
 $2C_2 = -5 \Rightarrow C_2 = -\frac{5}{2}$
The solution to the IVP is
 $y = 3 Cos(2x) - \frac{5}{2} Sin(2x)$.

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an nth order equation, we obtain an nth degree polynomial.
- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions e^{αx} cos(βx) and e^{αx} sin(βx).
- If a root *m* is repeated *k* times, we get *k* linearly independent solutions

$$e^{mx}$$
, xe^{mx} , x^2e^{mx} , ..., $x^{k-1}e^{mx}$

or in conjugate pairs cases 2k solutions

 $e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x), xe^{\alpha x}\cos(\beta x), xe^{\alpha x}\sin(\beta x), \dots,$

 $x^{k-1}e^{\alpha x}\cos(\beta x), \ x^{k-1}e^{\alpha x}\sin(\beta x)$

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It may require a computer algebra system to find the roots for a high degree polynomial.

Solve the ODE		stic equa	tion is
$y^{\prime\prime\prime}-4y^{\prime}=0$	LAGACHAN	she cjou	
3 de de	m ³ - 4		
factor	m (m² -)		
Every fund cauntal	m(m-2)	(m+2) = ()
Solution set must have 3 solutions.	M,= 0 ,	m2=2, M3	= -2
J	,=e ^{ox} = 1	zx Zz ⁼ e	-2x , y3 - e
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Note y= (, y, + 62 y2 + 63 y3.