March 1 Math 2306 sec 59 Spring 2016

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

We seek solutions of the form $y = e^{mx}$ for constant m, and obtain the characteristic (a.k.a. auxiliary) equation

$$am^2 + bm + c = 0.$$

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I Two distinct real roots $m_1 \neq m_2$, $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$.

II One repeated real root $m_1 = m_2 = m$, $y_1 = e^{mx}$ and $y_2 = xe^{mx}$.

III Two complex conjugate roots $m_{1,2} = \alpha \pm i\beta$, $y_1 = e^{\alpha x}\cos(\beta x)$ and $y_2 = e^{\alpha x}\sin(\beta x)$.

Example

Solve the IVP

$$y'' + 4y = 0, \quad y(0) = 3, \quad y'(0) = -5$$

Characteristic eqn. $m^2 + 4 = 0$

$m^2 = -4$

$m = \pm\sqrt{-4} = \pm 2i$

$\alpha \pm i\beta$

$\alpha = 0$

$\beta = 2$

$y_1 = e^{0x}\cos(2x) = \cos(2x)$

$y_2 = e^{0x}\sin(2x) = \sin(2x)$

The general solution is $y = c_1 \cos(2x) + c_2 \sin(2x)$

Apply $y(0) = 3$, $y'(0) = -5$

$y' = -2c_1 \sin(2x) + 2c_2 \cos(2x)$

$y(0) = c_1 \cos 0 + c_2 \sin 0 = 3 \implies c_1 = 3$

$y'(0) = -2c_1 \sin 0 + 2c_2 \cos 0 = -5 \implies 2c_2 = -5, \quad c_2 = \frac{-5}{2}$

The solution to the IVP is

$$\boxed{y = 3\cos(2x) - \frac{5}{2}\sin(2x)}$$

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.
- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$.
- If a root m is repeated k times, we get k linearly independent solutions

$$e^{mx},\quad xe^{mx},\quad x^2e^{mx},\quad \ldots,\quad x^{k-1}e^{mx}$$

or in conjugate pairs cases $2k$ solutions

$$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x),\quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x),\ldots,$$
$$x^{k-1}e^{\alpha x}\cos(\beta x),\ x^{k-1}e^{\alpha x}\sin(\beta x)$$

- It may require a computer algebra system to find the roots for a high degree polynomial.

Example

3rd order eqn. Every fundamental solution set must contain 3 functions.

Solve the ODE

$$y''' - 4y' = 0$$

Characteristic Eqn: $m^3 - 4m = 0$

factor $m(m^2 - 4) = 0$

$m(m-2)(m+2) = 0$ 3 different real roots

$m_1 = 0, \quad m_2 = 2, \quad m_3 = -2$

$y_1 = e^{0x} = 1, \quad y_2 = e^{2x}, \quad y_3 = e^{-2x}$

General solution $\qquad y = C_1 y_1 + C_2 y_2 + C_3 y_3$

$$y = C_1 + C_2 e^{2x} + C_3 e^{-2x}.$$

Example

Again 3rd order

Solve the ODE

$y''' - 3y'' + 3y' - y = 0$

Characteristic eqn.

$$m^3 - 3m^2 + 3m - 1 = 0$$

This is a perfect cube

$$(m-1)^3 = 0$$

$m = 1$ a triple root.

A fundamental solution set is

$$y_1 = e^x \quad , \quad y_2 = xe^x \quad , \quad y_3 = x^2 e^x$$

The general solution is

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$$

Example

Solve the ODE

$$\frac{d^4y}{dx^4}+2\frac{d^2y}{dx^2}+y=0$$

4th order equation. A fundamental solution set will contain 4 solutions.

Characteristic eqn:

$$m^4+2m^2+1=0$$

$$(m^2+1)^2=0$$

$$m=\pm i$$ each is a double root.

$$\alpha \pm i\beta \Rightarrow \alpha=0, \ \beta=1$$

Side note

$$m^2+1=0$$

$$m^2=-1$$

$$m=\pm i$$

We get

$$y_1 = e^{0x} \cos x = \cos x$$

$$y_2 = e^{0x} \sin x = \sin x$$

$$y_3 = x e^{0x} \cos x = x \cos x$$

$$y_4 = x e^{0x} \sin x = x \sin x$$

The general solution is

$$\boxed{y = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x.}$$

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example

Find a particular solution of the ODE

1st degree polynomial a.k.a a line.

$$y'' - 4y' + 4y = 8x + 1$$

Let's guess that y_p is a line (like the right hand side). A line has the form

$y_p = Ax + B$ $\quad$ A, B constants

See if this can solve the ODE.

$$y_p = Ax + B$$

$$y_p' = A$$

$$y_p'' = 0$$

Need

$$y_p'' - 4y_p' + 4y_p = 8x + 1$$

$$0 - 4 \cdot A + 4(Ax + B) = 8x + 1$$

$$-4A + 4Ax + 4B = 8x + 1$$

$$4Ax + (4B - 4A) = 8x + 1$$

This requires $4A = 8$ and

$$4B - 4A = 1$$

This holds if $A = 2$

$$4B = 1 + 4A = 1 + 8 \Rightarrow B = \frac{9}{4}$$

So $y_p = 2x + \frac{9}{4}$ is a particular solution.

The Method: Assume y_p has the same **form** as $g(x)$

$$y'' - 4y' + 4y = 6e^{3x}$$

To get derivatives to be constants times e^{3x},

we'll guess that $y_p = Ae^{3x}$ where A is constant.

$y_p = Ae^{3x}$

$y_p' = 3Ae^{3x}$

$y_p'' = 9Ae^{3x}$

we need

$$y_p'' - 4y_p' + 4y_p = 6e^{3x}$$

$$9Ae^{3x} - 4(3Ae^{3x}) + 4Ae^{3x} = 6e^{3x}$$

$$(9-12+4)Ae^{3x} = 6e^{3x}$$

$$Ae^{3x} = 6e^{3x}$$

This is true if $A=6$.

Hence $y_p = 6e^{3x}$ is a particular solution.

Make the form general

Constant times x^2 more generally it's a quadratic.

$$y'' - 4y' + 4y = 16x^2$$

Let's guess that $y_p = Ax^2$

$y_p = Ax^2$

$y_p' = 2Ax$

$y_p'' = 2A$

we need

$y_p'' - 4y_p' + 4y_p = 16x^2$

$2A - 4(2Ax) + 4Ax^2 = 16x^2$

$4Ax^2 - 8Ax + 2A = 16x^2$

Matching requires $4A=16$, $-8A=0$ and $2A=0$

This requires $A=4$ AND $A=0$.

Both can't be true at the same time.

Let's consider $g(x)=16x^2$ as a quadratic.

we'll guess that

$$y_p = Ax^2+Bx+C$$

$$y_p' = 2Ax+B$$

$$y_p'' = 2A$$

We require
$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax + B) + 4(Ax^2 + Bx + C) = 16x^2$$

$$4Ax^2 + (-8A + 4B)x + (2A - 4B + 4C) = 16x^2 + 0x + 0$$

Matching requires

$$4A = 16 \implies A = 4$$
$$-8A + 4B = 0$$
$$2A - 4B + 4C = 0$$

$$4B = 8A \Rightarrow B = 2A = 2\cdot 4 = 8$$

$$4C = -2A + 4B = -2\cdot 4 + 4\cdot 8 = -8 + 32 = 24$$

$$C = 6$$

So $y_p = 4x^2 + 8x + 6$ is a

particular solution.