

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- ▶ polynomials,
- ▶ exponentials,
- ▶ sines and/or cosines,
- ▶ and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

Constant coefficient left Polynomial right

Here $g(x) = 8x + 1$ a 1st degree polynomial.

Well "guess" that y_p is also a 1st degree polynomial.

Guess $y_p = Ax + B$ for constants A, B.

Sub into the ODE, it's supposed to solve it.

$$y_p' = A \quad y_p'' = 0$$

$$y_p'' - 4y_p' + 4y_p = 8x + 1$$

$$0 - 4(A) + 4(Ax + \beta) = 8x + 1$$

$$\underline{4Ax} + \underline{(-4A + 4\beta)} = \underline{8x + 1}$$

Match coefficients

$$4A = 8$$

$$-4A + 4\beta = 1$$

$$\Rightarrow A = 2$$
$$\beta = \frac{1}{4}(1 + 4A) = \frac{1}{4}(1 + 8) = \frac{9}{4}$$

We found the particular solution

$$y_p = 2x + \frac{9}{4}$$

The Method: Assume y_p has the same form as $g(x)$

$$y'' - 4y' + 4y = 6e^{-3x}$$

Constant coeff. left exponential right

$$g(x) = 6e^{-3x} \quad \text{a constant times } e^{-3x}$$

guess $y_p = Ae^{-3x}$, $y_p' = -3Ae^{-3x}$, $y_p'' = 9Ae^{-3x}$

Substitute $y_p'' - 4y_p' + 4y_p = 9Ae^{-3x} - 4(-3Ae^{-3x}) + 4Ae^{-3x} = 6e^{-3x}$

$$\Rightarrow 25Ae^{-3x} = 6e^{-3x} \Rightarrow A = \frac{6}{25}$$

So $y_p = \frac{6}{25}e^{-3x}$

Make the form general

$$y'' - 4y' + 4y = 16x^2$$

Suppose we consider $g(x) = 16x^2$ and identify it as "a constant times x^2 ." So we guess

$$y_p = Ax^2 \text{ . Substitute}$$

$$y_p' = 2Ax, \quad y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 2A - 4(2Ax) + 4Ax^2 = 16x^2$$

$$\underline{4Ax^2} - \underline{8Ax} + \underline{2A} = \underline{16x^2} + 0x$$

Match coefficients

$$\left. \begin{array}{l} 4A = 16 \\ -8A = 0 \end{array} \right\} \Rightarrow A=4 \text{ and } A=0$$

Not possible!

We need to identify $s(x) = 16x^2$ as a 2^{nd} degree polynomial

and set $y_p = Ax^2 + Bx + C$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 2A - 4(2Ax + B) + 4(Ax^2 + Bx + C) = 16x^2$$

$$\underline{4Ax^2} + \underline{(-8A + 4B)x} + \underline{(2A - 4B + 4C)} = \underline{16x^2} + \underline{0x} + \underline{0}$$

Matching

$$4A = 16 \Rightarrow A = 4$$

$$-8A + 4B = 0 \quad B = 2A = 8$$

$$2A - 4B + 4C = 0 \Rightarrow C = B - \frac{1}{2}A = 8 - 2 = 6$$

So $y_p = 4x^2 + 8x + 6$

General Form: sines and cosines

$$y'' - y' = 20 \sin(2x)$$

If we assume that $y_p = A \sin(2x)$, taking two derivatives would lead to the equation

$$-4A \sin(2x) - 2A \cos(2x) = 20 \sin(2x).$$

This would require (matching coefficients of sines and cosines)

$$-4A = 20 \quad \text{and} \quad -2A = 0.$$

This is impossible as it would require $-5 = 0!$

General Form: sines and cosines

We must think of our equation $y'' - y' = 20 \sin(2x)$ as

$$y'' - y' = 20 \sin(2x) + 0 \cos(2x).$$

The correct format for y_p is

$$y_p = A \sin(2x) + B \cos(2x).$$

Examples of Forms of y_p based on g (Trial Guesses)

(a) $g(x) = 1$ (or really any constant)
^
nonzero

$$y_p = A \quad \text{a constant function}$$

(b) $g(x) = x - 7$ 1st degree polynomial

$$y_p = Ax + B$$

Examples of Forms of y_p based on g (Trial Guesses)

(c) $g(x) = 5x^4$ 4th degree polynomial

$$y_p = Ax^4 + Bx^3 + Cx^2 + Dx + E$$

(d) $g(x) = 3x^3 - 5$ 3rd degree polynomial

$$y_p = Ax^3 + Bx^2 + Cx + D$$

Examples of Forms of y_p based on g (Trial Guesses)

(e) $g(x) = xe^{3x}$ 1st degree polynomial times e^{3x}

$$y_p = (Ax + B)e^{3x} = Axe^{3x} + Be^{3x}$$

(f) $g(x) = \cos(7x)$ Linear combination of $\cos(7x)$ and $\sin(7x)$

$$y_p = A \cos(7x) + B \sin(7x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(g) $g(x) = \sin(2x) - \cos(4x)$ Linear Combos of sine and cosine of both $2x$ and $4x$.

$$y_p = A \sin(2x) + B \cos(2x) + C \sin(4x) + D \cos(4x)$$

(h) $g(x) = x^2 \sin(3x)$ 2nd degree poly times $\sin(3x)$ plus 2nd degree poly times $\cos(3x)$

$$y_p = (Ax^2 + Bx + C) \sin(3x) + (Dx^2 + Ex + F) \cos(3x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(i) $g(x) = e^x \cos(2x)$

$$y_p = A e^x \cos(2x) + B e^x \sin(2x)$$

(j) $g(x) = x e^{-x} \sin(\pi x)$

$$y_p = (Ax + B) e^{-x} \sin(\pi x) + (Cx + D) e^{-x} \cos(\pi x)$$

The Superposition Principle

Example: Determine the correct form of the particular solution using the method of undetermined coefficients for the ODE

$$y'' - 4y' + 4y = 6e^{-3x} + 16x^2$$

By superposition, we can write $y_p = y_{p_1} + y_{p_2}$

where y_{p_1} solves

$$y'' - 4y' + 4y = 6e^{-3x}$$

$$y_{p_1} = Ae^{-3x}$$

where y_{p_2} solves $y'' - 4y' + 4y = 16x^2$

$$y_{p_2} = Bx^2 + Cx + D$$

The form of y_p : $y_p = Ae^{-3x} + Bx^2 + (x+D)$.

We found earlier $y_{p_1} = \frac{6}{25}e^{-3x}$

and $y_{p_2} = 4x^2 + 8x + 6$

$$y_p = \frac{6}{25}e^{-3x} + 4x^2 + 8x + 6$$

A Glitch!

$$y'' - y' = 3e^x$$

$g(x) = 3e^x$, so we guess $y_p = Ae^x$

Substitute $y_p = Ae^x$, $y_p' = Ae^x$

$$y_p'' - y_p' = Ae^x - Ae^x = 3e^x$$

$$0 = 3e^x$$

$$0 = 3 \mid \text{not possible.}$$

Consider the associated homogeneous eqn

$$y'' - y' = 0$$

Characteristic eqn

$$m^2 - m = 0$$

$$m=0$$

$$m(m-1) = 0 \Rightarrow$$

or

$$m=1$$

$$y_1 = e^{-x}, \quad y_2 = e^x$$

$$y_c = C_1 + C_2 e^x$$

We need to correct for when

g solves the homogeneous equation.